MAXPOLYNOMIALS AND MORPHOLOGICAL TEMPLATE DECOMPOSITION

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Image algebra and combinatorial optimization have led to the consideration of polynomials over lattice-ordered groups instead of over the usual structure of rings. These polynomials are referred to as maxpolynomials. Maxpolynomials were first introduced to solve combinatorial problems. This use is more restricted than their applications to image algebra. Therefore, a general development of the concepts related to maxpolynomials was in order.

A general definition of maxpolynomials is the starting point of this research. Maxpolynomials are defined for both the single-variable and the several-variable cases. These new definitions allow for the complete classification of maxpolynomials by way of a universal mapping property.

Past research in image algebra has established that maxpolynomial factorization is equivalent to morphological template decomposition. Several elements of maxpolynomial factorization are also investigated. First a division algorithm is demonstrated. From there, new factorization techniques are presented. Two lattice-ordered groups are the central focus of the factorization techniques. The first is built around the real numbers and negative infinity. This lattice-ordered group is used for gray-scale morphological templates. The second is built around just zero and negative infinity. Its applications are chiefly in binary morphology.

Another method of template decomposition is based on matrix analysis. A matrix decomposition algorithm utilizing nonlinear operations and the definition of rank in terms of minimax algebra is also developed.

CHAPTER 1
INTRODUCTION

The results presented here add to the expanding frontiers of image algebra. There are many specific examples of algebraic structures, and the power of the abstract point of view becomes apparent when results for an entire class are obtained by proving a single result for an abstract structure. This is the goal of image algebra.

The history of image algebra begins with mathematical morphology. The term morphology denotes a study of form. It is commonly used to describe a branch of biology which studies the structure of plants and animals. In image processing there is mathematical morphology. It is a tool which is used to rigorously quantify geometric structure or texture within an image. Mathematical morphology views the image as a collection of sets and then interprets how other sets interact with the image. It was developed in the mid 1960s by G. Matheron and J. Serra at the Paris School of Mines in Fontainbleau [1]. From a few basic operations they developed many different algorithms.

Two very important theorems about mathematical morphology were proved by Hadwiger and Matheron. In 1975, Matheron proved that any increasing mapping on $\mathbf{R}^n$ is both a union of erosions and an intersection of dilations [2]. Hadwiger showed that suitably well-behaved image functionals posses a similar property [3]. The beauty of morphology lies in these two theorems. They show that a wide class of operators can be represented by just a few morphological operations. Complete characterizations such as these are some of the most powerful theorems in mathematics. They generally serve to confirm a particular approach to a problem.

These observations led Serra and Sternberg to unify the concepts of morphology in hopes of bringing together many different aspects of image processing. Sternberg began to use the term image algebra to describe this unification [4].

Their attempt at generalization had a serious drawback. Many image operations are not expressible in morphological terms. Some transformations such as the Fourier transformation and histogram equalization are basic to digital image processing but cannot be accomplished using purely morphological methods. To remedy this shortcoming, G. X. Ritter set out to develop a universal system. The goals were to define a complete algebra which would encompass all image processing techniques, and to define a simple algebra whose operands and operators would be intelligible to those without an extensive mathematical background [5].

Once a comprehensive framework was built, the relationships between image algebra and other existing algebraic structures could be determined. This would turn out to be a prolific means of enhancing image understanding.

J.L. Davidson and H.J. Heijmans independently discovered that mathematical morphology could be formulated in terms of lattice algebra as well as the traditional set theoretic approach [6, 7]. Davidson's results further showed that morphology, with this reformulation, could be embedded into image algebra. They showed that morphological operations can be computed using lattice convolutions. In fact, lattice convolutions can do more that just morphology. The results further established the connection between mathematical morphology and minimax algebra. Lattice convolutions are based on minimax algebra.

Minimax theory has long been used to solve problems in operations research, such as machine scheduling and shortest-path problems. This theory is built around semilattice-ordered semigroups, also known as belts. A belt is set together with a lattice operation

and a binary operation which distributes over the lattice operation. It is typically denoted by $(\mathbf{F}, \vee, +)$. In much the same way that one investigates structures over rings, one also investigates homomorphisms, linear transformations, and matrices over belts. In fact, minimax problems for piecewise linear functions lead Cuninghame-Green and Miejer to develop the theory of maxpolynomials, which are polynomials over belts [8].

Maxpolynomials have the additional property that, in much the same way that polynomials can be used to calculate linear convolutions, they can be used to calculate lattice convolutions. However, a major drawback of the original development of maxpolynomials is that they were viewed as functional expressions. Unlike a polynomial, a maxpolynomial is quite different when viewed alternately as a formal expression and as a functional expression. For example, while it is true that for $x \in \mathbf{R}_{-\infty}$,

$$(2 + 2x) \vee (1 + x) \vee 1 = (2 + 2x) \vee 1,$$

formally they differ. When calculating lattice convolutions, maxpolynomials are taken to be formal expressions. All of the development given in this work will treat them as such.

The processing of images is a computationally intensive task. Convolutions require a large number of operations, which is proportional to not only the size of the image, but also to the size of the template. Template decomposition is one of the best ways to reduce the computational complexity of an algorithm.

In their initial investigation Cuninghame-Green and Meijer presented a factorization theorem for maxpolynomials. The necessary and sufficient condition for the application of their result is that the maxpolynomial be irredundant. This means that when viewed as a functional expression, it has no extraneous terms. In the above example, $(1 + x)$ on the left side of the equation is an extraneous term. Furthermore, when expanding their factorization it was only guaranteed that the result would be functionally the same as the

original. Li expanded their theorem to give conditions under which the original and the expansion of their factorization would be identical formal expressions [9].

The goal of this dissertation is to develop the theory of maxpolynomials beyond the work of Li [9], and Cuninghame-Green and Meijer [8]. By solidifying the foundation of maxpolynomials, we hope that they will become a generous resource for many applications. To insure the usefulness of the factoring techniques presented in this dissertation, all maxpolynomials are regarded as formal expressions. Hence, they are directly applicable to lattice convolutions.

In order to develop the theory of maxpolynomials, this dissertation begins with a review of some relevant minimax definitions. We present these axiomatics and basic manipulative properties in Chapter 2. The names and definitions for several types of belts are given. In addition, the concepts of homomorphism and duality are presented. These concepts form the basis of maxpolynomials analogously to the way in which ring theory is the basis of polynomial investigations.

Next, we present some of the foundations of image algebra. The focus of the third chapter is to show some of the ways in which minimax algebra and image algebra interact. The presentation is far from complete. However, it serves to familiarize the reader with the basic concepts.

Chapter 4 begins a rigorous establishment of maxpolynomials. First maxpolynomials are defined for a single indeterminate. Some elementary properties and notation are then developed. The construction of maxpolynomials in n indeterminates is next and is followed by some of their basic properties. In particular, we relate the structure of maxpolynomials back to the structures mentioned in Chapter 2. The main result of Chapter 4 is the complete classification of the belt of maxpolynomials using a universal mapping property.

In Chapter 5 we explore various concepts associated with factorization. Among the basic properties is the establishment of an analog to the division algorithm. From there, particular factorization theorems are presented for the two most common belts used in lattice convolutions.

Many of the considerations used in factoring maxpolynomials stem from those in the work of Z. Manseur and D. Wilson [10]. They used conditions such as symmetry and skew symmetry to aid in factoring polynomials. They also looked at how factoring boundary polynomials effected factorization.

Section 5.3 focuses on $(\mathbf{R}_{-\infty}, \vee, +)$, which is used for gray scale morphology. Several techniques for the single-variable situation are developed. Then the two principal techniques are applied to the two-variable case. The belt of Section 5.4 is $(\{-\infty, 0\}, \vee, +)$, which corresponds to binary image manipulations. The first part of the section shows that factoring by grouping arises in three important cases. It is then shown that when decomposing a binary restricted-convex template (see Section 5.4), only decompositions of the boundary need be considered. We then prove that the boundary involves only the three cases shown in the beginning. Once this is done, we have finished classifying the problem of decomposing restricted-convex templates.

The main focus of this dissertation is the development of the theory of maxpolynomials. Particular emphasis is placed on their use in morphological template deomposition. There are other methods used in morphological template decomposition. One of those methods is based on matrix analysis.

In the setting of linear algebra, D. O'Leary showed that if a 5×5 matrix has either rank 1 or all of its nonzero terms are on a single diagonal, then it can be factored into the product of two 3×3 matrices [11]. Z. Manseur and D. Wilson reduced the number of factors implied by O'Leary's result for the decomposition of an arbitrary matrix by using

polynomial methods [10]. J. Davidson studied some nonlinear matrix decompositions based on minimax algebra [12]. However, the work of Davidson did not utilize the rank of a matrix. The Goal of Section 6.3 is to prove a rank based decomposition in terms of minimax algebra.

CHAPTER 2
MINIMAX ALGEBRA

2.1 Introduction

When solving problems chiefly of interest to the operational researcher, a number of different authors discovered that these problems could be reformulated under a nonlinear algebraic structure. This reformulation presented a unifying language and thus a mutual strategy for solution. The language consists of an algebra. This algebra contains the extended real numbers and two binary operations. The two binary operations are maximum or minimum, and addition. We can denote this algebra by $(\mathbf{R}_{\pm\infty}, \vee, \wedge, +)$. Authors such as Giffler applied this structure to solve machine scheduling problems [13]. Others used it in shortest-path problems of graph theory [14, 15]. The properties of the lattice-ordered group $(\mathbf{R}_{\pm\infty}, \vee, \wedge, +)$ have been investigated over many years. However, the study of spaces of $n-$tuples over this algebra led to an elemental connection between operations research and linear algebra.

A unified account of this algebra and its connection to linear algebra was presented by Cuninghame-Green in his book Minimax Algebra [16].

J. Davidson showed that minimax algebra could be embedded into image algebra and that some of the basic results which had been obtained in the area of operations research have applications in image processing [6]. Although it had already been formally proven that image algebra was capable of representing any image transformation, the isomorphism that Davidson developed showed that minimax theory could be applied to image analysis.

The next section introduces some of the basic definitions and notation of minimax algebra. This presentation does not aim at completeness. Only those concepts which will be used are covered.

2.2 Belts

Let $\mathbf{F}$ be a set. We define on $\mathbf{F}$ two binary operations, $\vee$ and $*$, having the following properties.

1. Associativity of $\vee$: $x \vee (y \vee z) = (x \vee y) \vee z$.
2. Commutativity of $\vee$: $x \vee y = y \vee x$.
3. Idempotent: $x \vee x = x$.
4. Associativity of $*$: $x * (y * z) = (x * y) * z$.
5. Right distributive: $x * (y \vee z) = (x * y) \vee (x * z)$.
6. Left distributive: $(y \vee z) * x = (y * x) \vee (z * x)$.

The ordered triple $(\mathbf{F}, \vee, *)$ is known as a belt.

The properties 1–3 define a semilattice structure, that is, an abelian semigroup in which every element is idempotent. A semilattice is also referred to as a commutative band in some literature. It is the basis for what follows, similar to the way a group is the basis for the structure of a ring. In addition, the operation * is associative and satisfies "distributive" laws. Due to the similarity between this structure and a ring we call the structure $(\mathbf{F}, \vee, *)$ a belt. We also refer to $\vee$ as addition and * as multiplication. A belt is also known as a semilattice-ordered semigroup.

If we define $\vee$ to be the maximum of two numbers and $*$ to be the usual addition, then the set of real numbers with these operations, denoted by $(\mathbf{R}, \vee, +)$, is an example of a belt. Another example may be formed by taking the set $\mathbf{F}$ to be the positive real

numbers, $\mathbf{R}^+$, and binary operations to be maximum, $\vee$, and multiplication, $\times$. This belt is denoted by $(\mathbf{R}^+, \vee, \times)$.

Any semilattice may be viewed as a belt when the multiplication is defined to be identical to the semilattice operation. In this case, we say that the belt is a degenerate belt.

Let $(\mathbf{F}_1, \vee, *)$ and $(\mathbf{F}_2, \vee, *)$ be belts. A function $\psi : \mathbf{F}_1 \to \mathbf{F}_2$ is a belt homomorphism if

$$\psi(x \vee y) = \psi(x) \vee \psi(y)$$

and

$$\psi(x * y) = \psi(x) * \psi(y) .$$

Similarly, we use the terms isomorphism, endomorphism, and automorphism.

For example, if $\psi : \mathbf{R} \to \mathbf{R}^+$ is defined by

$$\psi(x) = e^x ,$$

then it is evident that $(\mathbf{R}, \vee, +)$ is isomorphic to $(\mathbf{R}^+, \vee, \times)$, where $\mathbf{R}^+$ is the positive real numbers.

A particular belt may also satisfy

7. Commtativity of $*$: $x * y = y * x$.

Such a belt is called a commutative belt.

If there exists an element $1_{\mathbf{F}}$ such that

8. Identity: $1_{\mathbf{F}} * x = x * 1_{\mathbf{F}} = x$,

then element 1_F is called the identity element and a belt satisfying axiom 8 is a belt with identity.

Suppose that for each $x \in \mathbf{F}$ there exists an element x' such that

9. Inverse: $x * x' = 1_{\mathbf{F}}$

Such an element is called the inverse of x.

It can be shown that the one-sided inverse of an element is its two-sided inverse and that the inverse is unique. We denote the inverse of an element x by x^{-1}. It is also evident that $\left(x^{-1}\right)^{-1} = x$ and $(1_{\mathbf{F}})^{-1} = 1_{\mathbf{F}}$. If a belt satisfies both axioms 8 and 9, then it is a division belt.

If there exists an element $-\infty$ such that

$$x \vee (-\infty) = x$$

and

$$-\infty * x = x * (-\infty) = -\infty ,$$

then such an element is unique and termed the null element. The existence of a null element is quite significant in the sequel. In fact, many of the derivations will depend on its presence. Fortunately an arbitrary nondegenerate belt may be extended to include a null element. The element $-\infty$ can be adjoined to the set $\mathbf{F}$ and this new set is denoted by $\mathbf{F}_{-\infty}$. This element serves as a lower bound for the semilattice. So we define

$$x \vee (-\infty) = (-\infty) \vee x = x .$$

The semigroup operation can be extended by defining

$$-\infty * x = x * (-\infty) = -\infty .$$

The elements of F which are different from $-\infty$ are called the finite elements of $\mathbf{F}$. It has been shown that, except in the trivial case, where $\mathbf{F}=\{1\}$ a partially ordered group cannot have universal bounds [17]. Thus a division belt cannot have a null element.

Notice that $(\mathbf{R}, \vee, +)$ is in fact a division belt. We may adjoin $-\infty$ to $\mathbf{R}$. It follows that $(\mathbf{R}_{-\infty}, \vee, +)$ is a belt with a null element in which the finite elements form a division belt. The belt $(\{-\infty, 0\}, \vee, +)$ may be considered as a subbelt of $(\mathbf{R}_{-\infty}, \vee, +)$. The belt $(\{-\infty, 0\}, \vee, +)$ is again a belt with a null element in which the finite elements form a division belt.

Under the mapping $\psi(x) = e^x$, the belt $(\mathbf{R}, \vee, +)$ can be shown to be isomorphic to $(\mathbf{R}^+, \vee, \times)$. If we extend the map ψ by defining $\psi(-\infty) \equiv 0$, then we have that $(\mathbf{R}_{-\infty}, \vee, +)$ is isomorphic to $(\mathbf{R}^{\geq 0}, \vee, \times)$, where $\mathbf{R}^{\geq 0}$ denotes all real numbers greater than or equal to zero. Note that the null element of $(\mathbf{R}^{\geq 0}, \vee, \times)$ is zero. By the uniqueness of the null element,

$$\left(\mathbf{R}^{\geq 0}_{-\infty}, \vee, \times\right) \equiv (\mathbf{R}^{\geq 0}, \vee, \times) .$$

The possibility exists to expand the structure of an arbitrary belt to include dual operations. That is, it would have the additional properties that for all $x, y, z \in \mathbf{F}$,

1'. Associativity of $\wedge$: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

2'. Commutativity of $\wedge$: $x \wedge y = y \wedge x$.

3'. Idempotent: $x \wedge x = x$.

4'. Associativity of $*'$: $x *' (y *' z) = (x *' y) *' z$.

5'. Right distributive: $x *' (y \wedge z) = (x *' y) \wedge (x *' z)$.

6'. Left distributive: $(y \wedge z) *' x = (y *' x) \wedge (z *' x)$.

If the two semilattice operations satisfy

10. Lattice absorption law: $x \vee (y \wedge x) = x \wedge (y \vee x) = x$,

then it is said that the two semilattice operations are consistent and that $(\mathbf{F}, \wedge, *')$ is the dual of $(\mathbf{F}, \vee, *)$ and vice versa. Thus, if it is possible to define these two additional

operations we say that $(\mathbf{F}, \vee, *)$ has duality. This is often represented by $(\mathbf{F}, \vee, \wedge, *, *')$. It is not assumed that $*$ and $*'$ are related. However, if they should coincide, then we say that the belt has a self dual multiplication. If $(\mathbf{F}_{-\infty}, \vee, *)$ is a division belt, then by defining $x \wedge y = \left(x^{-1} \vee y^{-1}\right)^{-1}$ we have introduced a dual semilattice operation and we get a division belt with self dual multiplication.

The belt $(\mathbf{R}, \vee, +)$ may be expanded by the inclusion of a minimum operation. It is easily checked that $(\mathbf{R}, \wedge, +)$ is the dual of $(\mathbf{R}, \vee, +)$.

Let $(\mathbf{F}_1, \vee, *)$ and $(\mathbf{F}_2, \vee, *)$ be belts. We shall say that $(\mathbf{F}_1, \vee, *)$ is conjugate to $(\mathbf{F}_2, \vee, *)$, if there exists a function $\psi : \mathbf{F}_1 \to \mathbf{F}_2$ such that

$$\psi \text{ is bijective;}$$

$$\text{for all } x, y \in \mathbf{F},\ \psi(x \vee y) = \psi(x) \wedge \psi(y);$$

$$\text{for all } x, y \in \mathbf{F},\ \psi(x * y) = \psi(x) *' \psi(y).$$

In particular, if $(\mathbf{F}, \vee, *)$ is a belt with duality, then we say that it is self-conjugate, if $(\mathbf{F}, \vee, *)$ is conjugate to $(\mathbf{F}, \wedge, *)$.

If $(\mathbf{F}, \vee, *)$ has a conjugate we denote by $(\mathbf{F}, \vee, *)^*$ the image of the conjugate map ψ. If $f \in \mathbf{F}$ we denote by $f^*\ \psi(f)$. We call f^* the conjugate of f It is immediate that $\left((\mathbf{F}, \vee, *)^*\right)^* = (\mathbf{F}, \vee, *)$ and $(f^*)^* = f$.

We note that every division belt is self conjugate under the map $f \to f^{-1}$. Unless otherwise noted, our reference to the dual of a given division belt shall be with respect to this mapping.

If we again consider the belt $(\mathbf{R}, \vee, +)$, then under the map $\psi(r) = -r$, we see that $(\mathbf{R}, \vee, +)$ is conjugate to $(\mathbf{R}, \wedge, +)$.

CHAPTER 3
IMAGE ALGEBRA

3.1 Introduction

Image algebra is a response to the need of the image processing community to have an axiomatic development of the field of image processing. In an axiomatic, or abstract, treatment of a given type of algebraic structure one assumes a small number of properties as axioms and then deduces many other properties from those axioms. Thus, it is possible to deal simultaneously with all the structures satisfying a given set of axioms instead of with each structure individually.

The term image algebra was first used by Sternberg to describe morphological operations [4]. Mathematical morphology is well suited for algebraic abstraction of its properties. Many of its techniques are expressible as combinations of simple operations. However, it lacked the generality to express many common image processing techniques. Techniques such as histogram equalization and image rotation are not expressible in terms of simple morphological operations.

The establishment of a general image algebra became the goal of G. X. Ritter at the University of Florida. Objects such as value sets and images were defined in general terms, with minimum specification. The result of Ritter's work has been shown to be capable of expressing all image processing operations [5].

J. Davidson showed that minimax algebra could be embedded into image algebra and that some of the basic results which had been obtained in the area of operations research have applications in image processing [6]. Although it had already been formally

proven that image algebra was capable of representing any image transformation, the isomorphism that Davidson developed showed that minimax theory could be applied to image analysis. In particular, the use of lattice convolutions showed how morphology is a subalgebra of image algebra.

Image algebra is a heterogeneous algebraic structure. That is, it consists of a number of different operands and operators. This chapter presents some of the basic concepts and notation of image algebra. Only those concepts which will be used in the sequel are reviewed. An in depth review may be found in Ritter et al. [18].

3.2 Basic Definitions

The value set is a homogeneous algebra. It is a set together with at least one binary operation. Generally, our interest will be concentrated on the set consisting of the real numbers along with negative infinity. Several different operations may be considered. We denote this by $\mathbf{R}_{-\infty}$. An arbitrary value set will be denoted by $\mathbf{F}$.

A spatial domain can be any topological space. Subsets of $\mathbf{R}^n$ will be our main focus with most applications being $\mathbf{Z}^n$. The symbol $\mathbf{Z}^n$ represents the n fold Cartesian product of the integers.

Let $\mathbf{X}$ be a spatial domain and $\mathbf{F}$ a value set. An $\mathbf{F}$ valued image on $\mathbf{X}$ is any map from $\mathbf{X}$ to $\mathbf{F}$. We denote the set of all $\mathbf{F}$ valued images on $\mathbf{X}$ by $\mathbf{F}^{\mathbf{X}}$.

We shall not distinguish between the graph of an image and the map. The graph of an image is also referred to as the data structure representation of the image. Given the data structure representation $\mathbf{a} = \{(\mathbf{x}, \mathbf{a}(\mathbf{x})) : \mathbf{x} \in \mathbf{X}\}$, then an element $(\mathbf{x}, \mathbf{a}(\mathbf{x}))$ of the data structure is called a picture element or pixel. The first coordinate, $\mathbf{x}$, of a pixel is called the pixel location or image point, and the second coordinate, $\mathbf{a}(\mathbf{x})$, is the pixel value or gray value of $\mathbf{a}$ at location $\mathbf{x}$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be spatial domains and $\mathbf{F}$ a value set. An $\mathbf{F}$ valued template from $\mathbf{Y}$ to $\mathbf{X}$ is a function $\mathrm{t} : \mathbf{Y} \to \mathbf{F}^{\mathbf{X}}$.

Thus, a template is an image whose pixel values are images. We denote the set of all $\mathbf{F}$ valued templates from $\mathbf{Y}$ to $\mathbf{X}$ by $\left(\mathbf{F}^{\mathbf{X}}\right)^{\mathbf{Y}}$. For notational convenience we define $\mathrm{t_y} \equiv \mathrm{t(y)}$. The pixel values, $\mathrm{t_y(x)}$, of the image $\mathrm{t_y}$ are called the weights of the template at the target point y.

If t is a real or complex valued template from $\mathbf{X}$ to $\mathbf{Y}$, then the support of t is defined as

$$S(\mathrm{t_y}) = \{\mathrm{x} \in \mathbf{X} \ : \ \mathrm{t_y(x)} \neq \mathbf{0}\} \ .$$

For extended real-valued templates we also define the following support at infinity,

$$S_{+\infty}(\mathrm{t_y}) = \{\mathrm{x} \in \mathbf{X} \ : \ \mathrm{t_y(x)} \neq +\infty\},$$

$$S_{-\infty}(\mathrm{t_y}) = \{\mathrm{x} \in \mathbf{X} \ : \ \mathrm{t_y(x)} \neq \ -\infty\} \ .$$

If $\mathbf{X}$ is a spatial domain with an operation $+$, then a template $\mathrm{t} \in \left(\mathbf{F}^{\mathbf{X}}\right)^{\mathbf{X}}$ is said to be translation invariant (with respect to the operation $+$) if and only if for each $\mathrm{x}, \mathrm{y}, \mathrm{x}+\mathrm{z}, \mathrm{y}+\mathrm{z} \in \mathbf{X}$ we have that $\mathrm{t_y(x)} = \mathrm{t_{y+z}(x+z)}$. Templates that are not translation invariant are called translation variant or simply variant. Often a translation invariant template can be represented pictorially. For example, let $\mathbf{X} = \mathbf{Z}^2$ and $\mathrm{y} = (x, y)$ be an arbitrary point of $\mathbf{X}$. Define $\mathrm{t} \in \left(\mathbf{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ by $\mathrm{t_y(y)} = 2$, $\mathrm{t_y}(x \pm 1. y) = 1$, $\mathrm{t_y}(x, y \pm 1) = 1$, $\mathrm{t_y}(x \pm 2. y) = 0$, $\mathrm{t_y}(x, y \pm 2) = 0$,

$\mathbf{t_y}(x-1, y-1) = \mathbf{t_y}(x+1, y-1) = \mathbf{t_y}(x+1, y+1) = \mathbf{t_y}(x-1, y+1) = 0$ and $\mathbf{t_y}(x, y) = -\infty$ otherwise. The representation of $\mathbf{t}$ is

$$\mathbf{t} = \begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & \\ 0 & 1 & 2 & 1 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array} .$$

3.3 Operations

The operations on and between $\mathbf{F}^{\mathbf{X}}$ are naturally derived from the algebraic structure of the value set $\mathbf{F}$. For example, if γ is a binary operation defined on $\mathbf{F}$, then γ induces a binary operation on $\mathbf{F}^{\mathbf{X}}$ defined as follows.

Let $\mathbf{a}, \mathbf{b} \in \mathbf{F}^{\mathbf{X}}$. Then

$$\mathbf{a}\gamma\mathbf{b} = \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \ : \ \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x})\gamma\mathbf{b}(\mathbf{x}), \ \mathbf{x} \in \mathbf{X}\} .$$

For an $\mathbf{F}$ valued image on a coordinate set $\mathbf{X}$ we have the following basic operations;

$$\mathbf{a} + \mathbf{b} \equiv \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \ : \ \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}), \ \mathbf{x} \in \mathbf{X}\}$$

$$\mathbf{a} * \mathbf{b} \equiv \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \ : \ \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) * \mathbf{b}(\mathbf{x}), \ \mathbf{x} \in \mathbf{X}\}$$

$$\mathbf{a} \vee \mathbf{b} \equiv \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \ : \ \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \vee \mathbf{b}(\mathbf{x}), \ \mathbf{x} \in \mathbf{X}\}$$

$$\mathbf{a} \wedge \mathbf{b} \equiv \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \ : \ \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \wedge \mathbf{b}(\mathbf{x}), \ \mathbf{x} \in \mathbf{X}\} .$$

Induced unary operations are defined in a similar fashion. Any unary operation $g : \mathbf{F} \to \mathbf{F}$ induces a unary operation $g : \mathbf{F}^{\mathbf{X}} \to \mathbf{F}^{\mathbf{X}}$ defined by

$$g(\mathbf{a}) = \{(\mathbf{x}, \mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = g(\mathbf{a}(\mathbf{x})),\ \mathbf{x} \in \mathbf{X}\}\,.$$

Let $\mathbf{F} = \mathbf{R}_{\pm\infty}$. The additive dual of $(\mathbf{R}_{\pm\infty}, \vee, +)$ is denoted by $(\mathbf{R}_{\pm\infty}, \wedge, +)$ and is determined by the map $r \to -r$. For $\mathbf{a} \in (\mathbf{R}_{\pm\infty})^{\mathbf{X}}$, the additive dual is defined by

$$\mathbf{a}^*(\mathbf{x}) = \begin{cases} -\mathbf{a}(\mathbf{x}) & \text{if } \mathbf{a}(\mathbf{x}) \in \mathbf{R} \\ -\infty & \text{if } \mathbf{a}(\mathbf{x}) = +\infty \\ +\infty & \text{if } \mathbf{a}(\mathbf{x}) = -\infty\,. \end{cases}$$

Similarly, if $\mathbf{a} \in (\mathrm{R}_{\approx}^{\geq 0})^{\mathbf{X}}$, then the multiplicative dual is defined by

$$\mathbf{a}^*(\mathbf{x}) = \begin{cases} 1/\mathbf{a}(\mathbf{x}) & \text{if } \mathbf{a}(\mathbf{x}) \in \mathbf{R} \\ 0 & \text{if } \mathbf{a}(\mathbf{x}) = +\infty \\ +\infty & \text{if } \mathbf{a}(\mathbf{x}) = -\infty\,. \end{cases}$$

Generalized convolutions are one of the most useful consequences of the concept of a heterogeneous image product. They provide rules for combining images with templates and templates with templates.

Let $\mathbf{F_1}, \mathbf{F_2}$ and $\mathbf{F_3}$ be three value sets, and suppose $\bigcirc : \mathbf{F_1} \times \mathbf{F_2} \to \mathbf{F_3}$ is a binary operation. If $\mathbf{a} \in \mathbf{F_1^X}$, $\mathbf{t} \in \left(\mathbf{F_2^X}\right)^{\mathbf{Y}}$ and γ is an associative binary operation on $\mathbf{F}$, then for each $\mathbf{y} \in \mathbf{Y}$ we have $\mathbf{t_y} \in \mathbf{F_1^X}$. Thus, $\mathbf{a} \bigcirc \mathbf{t_y} \in \mathbf{F^X}$ and $\Gamma(\mathbf{a} \bigcirc \mathbf{t_y}) \in \mathbf{F}$. It follows that the binary operations $\bigcirc$ and X induce a binary operation

$$\textcircled{?} : \mathbf{F_1^X} \times \left(\mathbf{F_2^X}\right)^{\mathbf{Y}} \to \mathbf{F_3^Y}$$

where

$$\mathbf{b} = \mathbf{a} \textcircled{?} \mathbf{t} \in \mathbf{F_3^Y}$$

is defined by

$$\mathbf{b}(\mathbf{y}) = \Gamma(\mathbf{a} \bigcirc \mathbf{t_y}) = \Gamma(\mathbf{a}(\mathbf{x}) \bigcirc \mathbf{t_y}(\mathbf{x}))\,.$$

The expression $\mathbf{a} \textcircled{\gamma} \mathbf{t}$ is called a generalized convolution or the right convolution product of $\mathbf{a}$ with $\mathbf{t}$.

Substitution of different value sets and specific binary operations for γ and $\bigcirc$ results in a wide variety of different image transforms. The main focus here will come from the belt $(\mathbf{R}_{\pm\infty}, \vee, +)$.

The bounded lattice ordered group $(\mathbf{R}_{\pm\infty}, \vee, \wedge, +, +')$ provides for two lattice convolutions,

$$\mathbf{b} = \mathbf{a} \boxtimes \mathbf{t}$$

where

$$\mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathbf{X} \cap S_{-\infty}(\mathbf{t}_\mathbf{y})} [\mathbf{a}(\mathbf{x}) + \mathbf{t}_\mathbf{y}(\mathbf{x})]$$

and

$$\mathbf{b} = \mathbf{a} \boxtimes \mathbf{t}$$

where

$$\mathbf{b}(\mathbf{y}) = \bigwedge_{\mathbf{x} \in \mathbf{X} \cap S_{-\infty}(\mathbf{t}_\mathbf{y})} [\mathbf{a}(\mathbf{x}) + \mathbf{t}_\mathbf{y}(\mathbf{x})] \,.$$

We designate $\boxtimes$ as the additive maximum and $\boxtimes$ as the additive minimum.

The bounded lattice ordered group $(\mathbf{R}_{\infty}^{\geq 0}, \vee, \wedge, \times, \times')$ provides for two lattice convolutions,

$$\mathbf{b} = \mathbf{a} \textcircled{\vee} \mathbf{t}$$

where

$$\mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathbf{X} \cap S_{-\infty}(\mathbf{t}_\mathbf{y})} [\mathbf{a}(\mathbf{x}) \times \mathbf{t}_\mathbf{y}(\mathbf{x})]$$

and

$$\mathbf{b} = \mathbf{a} \textcircled{\wedge} \mathbf{t}$$

where

$$\mathbf{b}(\mathbf{y}) = \bigwedge_{\mathbf{x}\in\mathbf{X}\cap S_{-\infty}(\mathbf{t}_\mathbf{y})} [\mathbf{a}(\mathbf{x}) \times \mathbf{t}_\mathbf{y}(\mathbf{x})] .$$

We designate $\bigvee$ as the multiplicative maximum and $\bigwedge$ as the multiplicative minimum.

The common unary and binary operation on templates correspond to those defined on images. For example, if $g : \mathbf{F_1} \to \mathbf{F_2}$ and $\mathbf{t} \in \left(\mathbf{F}_1^{\mathbf{X}}\right)^{\mathbf{Y}}$, then $\mathbf{r} = g \circ \mathbf{t} \in \left(\mathbf{F}_2^{\mathbf{X}}\right)^{\mathbf{Y}}$ is defined by

$$\mathbf{r}_\mathbf{y} = g(\mathbf{t}_\mathbf{y}) ,$$

where g is applied point-wise to the image $\mathbf{t}_\mathbf{y}$.

Let $\mathbf{t} \in \left(\mathbf{F}^{\mathbf{X}}\right)^{\mathbf{Y}}$. The transpose of $\mathbf{t}$ is a template $\mathbf{t}' \in \left(\mathbf{F}^{\mathbf{Y}}\right)^{\mathbf{X}}$ defined by $\mathbf{t}'_\mathbf{y}(\mathbf{x}) \equiv \mathbf{t}_\mathbf{x}(\mathbf{y})$.

For $\mathbf{t} \in \left(\left(\mathbf{R}_{\pm\infty}\right)^{\mathbf{X}}\right)^{\mathbf{Y}}$, the additive dual of $\mathbf{t}$ is the template $\mathbf{t}$ defined by

$$\mathbf{t}_\mathbf{x}(\mathbf{y}) = \begin{cases} -\mathbf{t}_\mathbf{y}(\mathbf{x}) & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) \in \mathbf{R} \\ -\infty & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) = +\infty \\ +\infty & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) = -\infty . \end{cases}$$

For $\mathbf{t} \in \left(\left(\mathbf{R}_{\infty}^{\geq 0}\right)^{\mathbf{X}}\right)^{\mathbf{Y}}$, the multiplicative dual of $\mathbf{t}$ is the template $\mathbf{t}$ defined by

$$\mathbf{t}_\mathbf{x}(\mathbf{y}) = \begin{cases} 1/\mathbf{t}_\mathbf{y}(\mathbf{x}) & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) \in \mathbf{R} \\ -\infty & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) = +\infty \\ +\infty & \textbf{if } \mathbf{t}_\mathbf{y}(\mathbf{x}) = -\infty . \end{cases}$$

We saw previously how two binary operations, γ and $\bigcirc$, could be combined to induce a convolution operator. This notion extends to templates as well.

Suppose that $\mathbf{s} \in \left(\mathbf{F}_1^{\mathbf{Z}}\right)^{\mathbf{X}}, \mathbf{t} \in \left(\mathbf{F}_2^{\mathbf{X}}\right)^{\mathbf{Y}}, \bigcirc : \mathbf{F}_1 \times \mathbf{F}_2 \to \mathbf{F}_3$, $(\mathbf{F}_3, \gamma)$ a commutative semigroup, and $\mathbf{X}$ a finite point set. The generalized convolution product $\mathbf{r} = \mathbf{s} \textcircled{\gamma} \mathbf{t}$, where $\mathbf{r} \in \left(\mathbf{F}_3^{\mathbf{X}}\right)^{\mathbf{Y}}$, is defined as

$$\mathbf{r}_\mathbf{y}(\mathbf{z}) = \gamma^{\mathbf{l}}(\mathbf{s}_\mathbf{x}(\mathbf{z}) \bigcirc \mathbf{t}_\mathbf{y}(\mathbf{x})) .$$

Let $\mathbf{t} \in \left(\mathbf{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{Y}}$ and $\mathbf{s} \in \left(\mathbf{R}_{-\infty}^{\mathbf{W}}\right)^{\mathbf{X}}$. Then $\mathbf{r} = \mathbf{s} \boxtimes \mathbf{t}$ is defined by the formula

$$\mathbf{r}_{\mathbf{y}}(\mathbf{z}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \left[\mathbf{s}_{\mathbf{x}}(\mathbf{z}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x})\right].$$

If $\mathbf{t} \in \left(\left(\mathbf{R}_{\infty}^{\geq \mathbf{0}}\right)^{\mathbf{X}}\right)^{\mathbf{Y}}$ and $\mathbf{t} \in \left(\left(\mathbf{R}_{\infty}^{\geq \mathbf{0}}\right)^{\mathbf{W}}\right)^{\mathbf{X}}$, then $\mathbf{r} = \mathbf{s} \circledcirc \mathbf{t}$ is defined by the formula

$$\mathbf{r}_{\mathbf{y}}(\mathbf{z}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \left[\mathbf{s}_{\mathbf{x}}(\mathbf{z}) \times \mathbf{t}_{\mathbf{y}}(\mathbf{x})\right].$$

Many other image and template operations are described in Ritter et al. [18].

In the subsequent discussion, we assume that $\mathbf{X} = \mathbf{Z}^2$, and $\mathbf{t} \in \left(\mathbf{F}^{\mathbf{X}}\right)^{\mathbf{X}}$ is a shift-invariant template with finite support at a point $\mathbf{y} \in \mathbf{X}$. If $\mathbf{x} = (x, y) \in \mathbf{X}$, then define $p_1(\mathbf{x}) = x$ and $p_2(\mathbf{x}) = y$. We have then that $S_{-\infty}(\mathbf{t_y})$ is finite and the following are well defined,

$$i(\mathbf{y})_{min} = \inf\{[p_1(\mathbf{x}) : \mathbf{x} \in S_{-\infty}(\mathbf{t_y})]\}, \quad i(\mathbf{y})_{max} = \sup\{[p_1(\mathbf{x}) : \mathbf{x} \in S_{-\infty}(\mathbf{t_y})]\},$$

$$j(\mathbf{y})_{min} = \inf\{[p_2(\mathbf{x}) : \mathbf{x} \in S_{-\infty}(\mathbf{t_y})]\}, \quad j(\mathbf{y})_{max} = \sup\{[p_2(\mathbf{x}) : \mathbf{x} \in S_{-\infty}(\mathbf{t_y})]\}.$$

Let

$$m(\mathbf{y}) = i(\mathbf{y})_{max} - i(\mathbf{y})_{min},$$

$$n(\mathbf{y}) = j(\mathbf{y})_{max} - j(\mathbf{y})_{min},$$

and define

$$R(\mathbf{t_y}) = \{(i(\mathbf{y})_{min} + i,\ j(\mathbf{y})_{min} + j) : 0 \leq i \leq m(\mathbf{y}),\ 0 \leq j \leq n(\mathbf{y}),\ i, j \in \mathbf{N}\}.$$

By definition $R(\mathbf{t_y})$ is a rectangular array, and it is the smallest rectangular array containing $S_{-\infty}(\mathbf{t_y})$.

A template, $\mathbf{t}$, with finite support is called a *rectangular $m \times n$ template* if $R(\mathbf{t_y})$ is of size $m \times n$.

Example. Let a morphological template, $\mathbf{t}$, is given by

$$\mathbf{t} = \begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & \\ 0 & 1 & 2 & 1 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array} .$$

The set $R(\mathbf{t_y})$ is given by

The diamond designates the origin.

CHAPTER 4
MAXPOLYNOMIALS

4.1 Introduction

The algebraic structure of a belt can be applied to the solution of minimax problems for piecewise linear functions. Cuninghame-Green and Meijer noted that certain combinatorial problems can be expressed using maxpolynomials [8]. These problems involve using maxpolynomials as functional expressions.

Maxpolynomials have a different use when they are considered as formal expressions. One use is the calculation of lattice convolutions such as $\boxtimes$ or $\circledvee$. To illustrate the similarities and differences between linear convolutions and lattice convolutions, suppose that two finite, discrete, one-dimensional signals are given. These signals may be regarded as functions from the set of integers into some set, the real numbers for example. Their convolution results in a finite, discrete signal and so it is also a polynomial. Let

$$f = a_0 + a_1 x + \cdots + a_n x^n$$

$$g = b_0 + b_1 x + \cdots + b_m x^m$$

and

$$f * g = c_0 + c_1 x + \cdots + c_{m+n} x^{m+n} .$$

The coefficients of the polynomials are the discrete values of the signal. The powers of the variable x serve to preserve the order of the coefficients.

The convolution of f and g is given by

$$(f * g)(j) = \sum_m f(m)g(j-m) \text{ for } j = 1, 2, \ldots .$$

If f, g and $f * g$ are replaced with their polynomial representations the convolution formula becomes

$$c_j = \sum_m a_m b_{j-m} .$$

Taking into account were the coefficients of f and g are nonzero, the formula reduces to

$$c_j = \sum_{m=0}^{j} a_m b_{j-m} ,$$

which is just the product of the polynomials.

Image algebra has the capability to represent generalized convolutions. These are convolutions where different binary operations are used, instead of the usual operations of addition and multiplication. For example, there is the generalized convolution called the additive maximum. The additive maximum of two finite, discrete, one-dimensional signals is represented in image algebra as $\boxtimes$ and is calculated by the formula

$$(f \boxtimes g)(j) = \bigvee_m (f(m) + g(j-m)) .$$

One may now be led to believe that it is possible to define a certain kind of "polynomial" whose product corresponds to this convolution. In the linear convolution we had

$$f = a_0 + (a_1 x) + (a_2 x^2) + (a_3 x^3) + \cdots .$$

The two operations were addition and multiplication. In a lattice convolution, the two operations are maximum and addition. To separate the coefficients we will now use

$\vee$, and to preserve the order of the coefficients instead of powers of a variable we use multiples and write

$$f = a_0 \vee (a_1 + x) \vee (a_2 + 2x) \vee (a_3 + 3x) \vee \cdots .$$

Dong Li noted the connection between maxpolynomials and the additive maximum convolution [9]. All of these observations may be extended to signals in two (images) or more dimensions.

The aim of this chapter is to classify maxpolynomials. That is to say that they will be identified as a member of an algebraic structure. By doing so, any investigation is not limited to the specific, and other results may be applied to this new member.

4.2 Basic Definitions

Let $(\mathbf{F}_{-\infty}, \vee, *)$ be a belt with lower bound $-\infty$.

Definition. All sequences of elements of $\mathbf{F}$ which have only finitely many elements which are not negative infinity are called *maxpolynomials over* $\mathbf{F}$.

The set of maxpolynomials over $\mathbf{F}$ is denoted by $\mathbf{F}_{-\infty}[x]$.

Theorem 4.2.1. *Let* $(\mathbf{F}_{-\infty}, \vee, *)$ *be a belt.*

(i) $\mathbf{F}_{-\infty}[x]$ *is a belt with* $\vee$ *and* $*$ *defined by*

$$(a_0, a_1, \ldots) \vee (b_0, b_1, \ldots) = (a_0 \vee b_0, a_1 \vee b_1, \ldots)$$

and

$$(a_0, a_1 \ldots) * (b_0, b_1, \ldots) = (c_0, c_1, \ldots)$$

where

$$c_n = \bigvee_{i=0}^{n} (a_{n-i} * b_i).$$

(ii) If $(\mathbf{F}_{-\infty}, \vee, *)$ *is a commutative belt [resp. a belt with identity], then so is* $\mathbf{F}_{-\infty}[x]$.

(iii) The map $\mathbf{F} \to \mathbf{F}_{-\infty}[x]$ *given by* $\psi(f) = (f, -\infty, -\infty, -\infty, ...)$ *is a monomorphism of belts.*

Proof: If $a, b, c \in \mathbf{F}_{-\infty}$, then $a = (a_0, a_1, ...)$, $b = (b_0, b_1, ...)$, and $c = (c_0, c_1, ...)$,

$$\begin{aligned} a \vee (b \vee c) &= a \vee (b_0 \vee c_0, b_1 \vee c_1, ...) \\ &= (a_0 \vee b_0 \vee c_0,\ a_1 \vee b_1 \vee c_1, ...) \\ &= (a_0 \vee b_0, a_1 \vee b_1, ...) \vee c \\ &= (a \vee b) \vee c \end{aligned}$$

$$\begin{aligned} a \vee b &= (a_0 \vee b_0, a_1 \vee b_1, ...) \\ &= (b_0 \vee a_0, b_1 \vee a_1, ...) \\ &= b \vee a \end{aligned}$$

$$\begin{aligned} a \vee a &= (a_0 \vee a_0, a_1 \vee a_1, ...) \\ &= (a_0, a_1, ...) \\ &= a \end{aligned}$$

$$
\begin{aligned}
(a * b) * c &= \bigvee_{j=0}^{n} \left(\bigvee_{i=0}^{n-j} a_i * b_{n-j-i} \right) * c_j \\
&= \bigvee_{j=0}^{n} \bigvee_{i=0}^{n-j} a_i * b_{n-j-i} * c_j \\
&= \bigvee_{i=0}^{n} \bigvee_{j=0}^{n-i} a_i * b_{n-j-i} * c_j \\
&= \bigvee_{i=0}^{n} a_i * \left(\bigvee_{j=0}^{n-i} b_{n-j-i} * c_j \right) \\
&= a * (b * c) .
\end{aligned}
$$

Let d_n be the nth coefficient of $a * (b \vee c)$. By calculation

$$
\begin{aligned}
d_n &= \bigvee_{i=0}^{n} a_{n-i}(b_i \vee c_i) \\
&= \bigvee_{i=0}^{n} (a_{n-i} * b_i) \vee (a_{n-i} * c_i) \\
&= \bigvee_{i=0}^{n} (a_{n-i} * b_i) \vee \bigvee_{i=0}^{n} (a_{n-i} * c_i) .
\end{aligned}
$$

Hence, d_n is also the nth coefficient of $(a * b) \vee (a * c)$. So

$$a * (b \vee c) = (a \vee b) * (a \vee c).$$

Next, let d_n be the nth coefficient of $(b \vee c) * a$. Again by calculation

$$
\begin{aligned}
d_n &= \bigvee_{i=0}^{n} (b_{n-i} \vee c_{n-i}) * a_i \\
&= \bigvee_{i=0}^{n} (a_i * b_{n-i}) \vee (a_i * c_{n-i}) \\
&= \left[\bigvee_{i=0}^{n} (a_i * b_{n-i}) \right] \vee \left[\bigvee_{i=0}^{n} (a_i * c_{n-i}) \right].
\end{aligned}
$$

Hence, d_n is the nth coefficient of $(b * a) \vee (c * a)$. So

$$(b \vee c) * a = (b \vee a) * (c \vee a).$$

If $\mathbf{F}_{-\infty}$ is commutative, then

$$\begin{aligned} a * b &= \bigvee_{i=0}^{n} a_i * b_{n-i} \\ &= \bigvee_{i=0}^{n} b_{n-i} * a_i \\ &= b * a \,. \end{aligned}$$

which shows that $\mathbf{F}_{-\infty}[x]$ is also commutative.

If $\mathbf{F}_{-\infty}$ has an identity $1_{\mathbf{F}}$, then the element $(1_{\mathbf{F}}, -\infty, -\infty, ...) \in \mathbf{F}_{-\infty}[x]$ acts as an identity in $\mathbf{F}_{-\infty}[x]$. By calculation,

$$(1_{\mathbf{F}},\ -\infty,\ -\infty, ...) * (a_0, a_1, a_2, ...) = (a_0, a_1, a_2, ...) \,.$$

To show that the mapping is a belt monomorphism, let $f_1, f_2 \in \mathbf{F}$. It follows that

$$\psi(f_1 \vee f_2) =$$

$$(f_1 \vee f_2, -\infty, -\infty, ...) = (f_1, -\infty, -\infty, ...) \vee (f_2, -\infty, -\infty, ...)$$

and

$$\psi(f_1 * f_2) =$$

$$(f_1 * f_2, -\infty, -\infty, ...) = (f_1, -\infty, -\infty, ...) * (f_2, -\infty, -\infty, ...) \,.$$

So the map is a belt homomorphism.

Suppose that

$$(f_1,\ -\infty,\ -\infty, ...) = (f_2,\ -\infty,\ -\infty, ...),$$

then clearly $f_1 = f_2$. So the map is also a monomorphism.

Q.E.D.

In view of part *ii* of the previous theorem, $\mathbf{F}_{-\infty}$ may be identified with its isomorphic image in $\mathbf{F}_{-\infty}[x]$ and we will write $(f, -\infty, -\infty, ...)$ as simply f. By calculation, we have that $f * (a_0, a_1, ...) = (f * a_0, f * a_1, ...)$.

The next theorem develops a notation which makes the connection between polynomials and maxpolynomials easier to see.

Theorem 4.2.2. *Let* $(\mathbf{F}_{-\infty}, \vee, *)$ *be a belt with identity and denote by* x *the element* $(-\infty, 1_{\mathbf{F}}, -\infty, -\infty. ...)$ *of* $\mathbf{F}_{-\infty}[\mathbf{x}]$.

(i) $nx = (-\infty, -\infty, ..., 1_{\mathbf{F}}, -\infty, ...)$, *where* $1_{\mathbf{F}}$ *is in the* $(n+1)$*st coordinate.*

(ii) If $f \in \mathbf{F}_{-\infty}$, *then for each* $n \geq 0$, $f * nx = nx * f = (-\infty, ..., -\infty, f. -\infty, ...)$, *where* f *is in the* $(n+1)$*st coordinate.*

(iii) For every non negative infinity maxpolynomial (that is a maxpolynomial with some element which is not $-\infty$*) in* $\mathbf{F}_{-\infty}[x]$, *there exists an integer* n *and elements* $a_0, a_1, ..., a_n \in \mathbf{F}_{-\infty}$ *such that* $g = a_0 \vee (a_1 * x) \vee ... \vee (a_n * nx)$. *The integer* n *and the elements* a_i *are unique.*

Proof: (i) By definition, the formula is true for $n = 1$. Suppose that $(n-1)x = (-\infty, -\infty, ..., 1_{\mathbf{F}}, -\infty, ...)$, where $1_{\mathbf{F}}$ is in the $(n-1)$–th coordinate. It follows that

$$nx = x + (n-1)x = (-\infty, 1_{\mathbf{F}}, -\infty, -\infty, ...) + (-\infty, ..., -\infty, 1_{\mathbf{F}}, -\infty, ...)$$

$$= (c_0, c_1, ...) .$$

If $j = n$, *then* $c_j = 1_{\mathbf{F}} * 1_{\mathbf{F}} = 1_{\mathbf{F}}$. If $j \neq n$, then $c_j = -\infty$.

(ii) $f * nx = (f, -\infty, -\infty, ...) * (-\infty, ..., -\infty, 0_{\mathbf{F}}, -\infty, ...)$. Straightforward computation show that $(f, -\infty, -\infty, ...) * (-\infty, ..., -\infty, 1_{\mathbf{F}}, -\infty, ...) = (-\infty, ..., -\infty, f. -\infty, ...)$. Similarly, for $nx * f$.

(iii) If $g = (a_0, a_1, ...)$, there must be a largest index n such that $a_n \neq -\infty$. It follows that $a_0, a_1, ..., a_n \in \mathbf{F}_{-\infty}$ are the desired elements. If $g = b_0 \vee (b_1 + x) \vee \cdots \vee (b_m + mx)$, then

$$(b_0, b_1, ..., b_m, -\infty, -\infty, ...) = (a_0, a_1, ..., a_n, -\infty, -\infty, ...)$$

and $a_i = b_i$.

Q.E.D.

If $\mathbf{F}$ has an identity, then $0x = 1_{\mathbf{F}}$ and we may write the maxpolynomial $a_0 * 0x \vee (a_1 * 1x) \vee \cdots \vee (a_n * nx)$ as $a_0 \vee (a_1 * x) \vee \cdots \vee (a_n * nx)$. An important difference between the two cases is that when there is an identity element, x is an element of the belt $\mathbf{F}_{-\infty}[x]$. Hereafter, a maxpolynomial f over a belt with identity will always be written in the form $f = a_0 \vee (a_1 * x) \vee \cdots \vee (a_n * nx)$. In this notation, maximum and addition are given by the following analogs to the familiar rules,

$$\bigvee_{i=0}^{n} (a_i * ix) \vee \bigvee_{i=0}^{n} (b_i * ix) = \bigvee_{i=0}^{n} ((a_i \vee b_i) * ix)$$

$$\left(\bigvee_{i=0}^{n} (a_i * ix) \right) + \left(\bigvee_{j=0}^{m} (b_j * jx) \right) = \bigvee_{\iota=0}^{m+n} (c_k * kx), \text{ where } c_k = \bigvee_{i+j=k} (a_i * b_j).$$

If $P = \bigvee_{i=0}^{n} (a_i * ix) \in \mathbf{F}_{-\infty}[x]$, then the elements a_i are called the coefficients of P. The coefficient a_0 is called the constant term. Elements of $\mathbf{F}_{-\infty}$, which all have the form $f = (f, -\infty, -\infty, ...)$, are called the constant maxpolynomials. If $P = \bigvee_{i=0}^{n} (a_i * ix)$ has $a_n \neq -\infty$, then a_n is called the leading coefficient. If $\mathbf{F}_{-\infty}$ has an identity and the leading coefficient of P is $1_{\mathbf{F}}$, then P is said to be a monic maxpolynomial. It shall be the convention here that when writing $P = \bigvee_{i=0}^{n} (a_i * ix)$, we have $a_n \neq -\infty$.

The next step is to define maxpolynomials in several variables. The starting point is that a sequence is a function defined on the Natural numbers. Let $\mathbf{N}$ be the Natural numbers and $\mathbf{N}^n = \mathbf{N} \times \mathbf{N} \times \cdots \times \mathbf{N}$ (n factors).

Theorem 4.2.3. *Let $(\mathbf{F}_{-\infty}, \vee, *)$ be a belt and denote by $\mathbf{F}_{-\infty}[x_1, ..., x_n]$ the set of all functions $g : \mathbf{N}^n \to \mathbf{F}_{-\infty}$ such that $g(u) \neq -\infty$ for at most a finite number of elements u of $\mathbf{N}^n$.*

(i) $\mathbf{F}_{-\infty}[x_1, ..., x_n]$ is a belt with $\vee$ and $*$ defined by

$$(g \vee h)(u) = g(u) \vee h(u)$$

and

$$(g * h)(u) = \bigvee_{v+w=u} g(v) * h(w) .$$

(ii) If $\mathbf{F}_{-\infty}$ is commutative (resp. a belt with identity), then so is $\mathbf{F}_{-\infty}[x_1, ..., x_n]$.

(iii) The map

$$\psi : \mathbf{F}_{-\infty} \to \mathbf{F}_{-\infty}[x_1, ..., x_n] ,$$

given by $\psi(f) = g_f$, where $g_f(0, ..., 0) = f$ and

$$g_f(u) = -\infty$$

for all other $u \in \mathbf{N}^n$, is a monomorphism of belts.

Proof: (i)

$$(h \vee g)(u) =$$

$$= h(u) \vee g(u)$$

$$= g(u) \vee h(u)$$

$$= (g \vee h)(u)$$

$$[(f \vee g) \vee h](u) =$$

$$= (f \vee g)(u) \vee h(u)$$

$$= f(u) \vee g(u) \vee h(u)$$

$$= f(u) \vee (g(u) \vee h(u))$$

$$= [f \vee (g \vee h)](u)$$

$$(g \vee g)(u) = g(u) \vee g(u) = g(u)$$

$$((f * g) * h) =$$

$$= \bigvee_{v+w=u} f(v) + \left(\bigvee_{y+z=w} g(y) * h(z) \right)$$

$$= \bigvee_{v+w=u} \bigvee_{y+z=w} f(v) * g(y) * h(z)$$

$$= \bigvee_{w+z=u} \bigvee_{v+y=w} f(v) * g(y) * h(z)$$

$$= \bigvee_{w+z=u} \left(\bigvee_{v+y=w} f(v) * g(y) \right) * h(z).$$

$$g * (h \vee s) =$$

$$g * (h \vee s)(u) = g(u) * (h \vee s)(u)$$

$$= g(u) * (h(u) \vee s(u))$$

$$= (g(u) * h(u)) \vee (g(u) * s(u))$$

$$= (g * h) \vee (g * s).$$

$$(h \vee s) * g =$$

$$[(h \vee s) * g](u) = [(h \vee s)](u) * g(u)$$

$$= (h(u) \vee s(u)) * g(u)$$

$$= (h(u) * g(u)) \vee (s(u) * g(u))$$

$$= (h * g) \vee (s * g)$$

(ii)

$$(g * h)(u) = \bigvee_{v+w=u} g(v) * h(w)$$

$$= \bigvee_{v+w=u} h(w) * g(v)$$

$$= (h * g)(u) \ .$$

Let $1_{\mathbf{F}}$ be the identity of $\mathbf{F}$. Define $\mathrm{I} : N^n \to \mathbf{F}_{-\infty}$ by $\mathrm{I}(u) = 1_{\mathbf{F}}$ if $u = (0, 0, 0, ..., 0)$. and $\mathrm{I}(u) = -\infty$ otherwise. We have then

$$(g * \mathrm{I})(u) = g(u) * \mathrm{I}(u)$$

$$= \bigvee_{v+w=u} g(v)\mathrm{I}(w)$$

$$= g(u - (0, 0, ..., 0)) * \mathrm{I}(0, 0, ..., 0)$$

$$= g(u) * 1_{\mathbf{F}}$$

$$= g(u) \ .$$

(iii) First, $g(f_1 \vee f_2) = g_{f_1 \vee f_2}$.

If $u = (0, ..., 0)$, then

$$g_{f_1 \vee f_2}(u) = f_1 \vee f_2 = g_{f_1}(u) \vee g_{f_2}(u).$$

If $u \neq (0, ..., 0)$, then

$$g_{f_1 \vee f_2}(u) = -\infty = g_{f_1}(u) \vee g_{f_2}(u).$$

Next, $\psi(f_1 * f_2) = g_{f_1 * f_2}$.

If $u = (0, ..., 0)$, then in order for $v + w = u$, it must be that $v = (0, ..., 0)$ and $w = (0, ..., 0)$ simultaneously. So,

$$\begin{aligned}(g_{f_1} * g_{f_2})(u) &= \bigvee_{v+w=u} g_{f_1}(v) * g_{f_2}(w) \\ &= g_{f_1}(0, ..., 0) * g_{f_2}(0, ..., 0) \\ &= f_1 * f_2 \\ &= g_{f_1 * f_2}(u) .\end{aligned}$$

If $u \neq (0, ..., 0)$, then it is not possible to have $v = (0, ..., 0)$ and $w = (0, ..., 0)$ simultaneously. Hence,

$$\begin{aligned}(g_{f_1} * g_{f_2})(u) &= \bigvee_{v+w=u} g_{f_1}(v) * g_{f_2}(w) \\ &= -\infty \\ &= g_{f_1 * f_2}(u) .\end{aligned}$$

So, ψ is a homomorphism.

If $\psi(f_1) = \psi(f_2)$, then $g_{f_1}(u) = g_{f_2}(u)$ for all u. In particular, if $u = (0, ..., 0)$, we see that $f_1 = f_2$.

Q.E.D.

The belt of the previous theorem is called the belt of *maxpolynomials in n indeterminates over $\mathbf{F}_{-\infty}$*. If $n = 1$, then $\mathbf{F}_{-\infty}[x]$ is the belt of maxpolynomials. As in the previous case, there is a more familiar notation.

Let n be a positive number and for each $i = 1, 2, ..., n$ let

$$\varepsilon_i = (0, ..., 0, 1, 0, ...0) \in \mathbf{N}^n,$$

where 1 is in the ith coordinate of ε_i. If $k \in \mathbf{N}$, let $k\varepsilon_i = (0, ..., 0, k, 0, ..., 0)$, then every element of $\mathbf{N}^n$ may be written in the form $k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_n\varepsilon_n$.

Theorem 4.2.4. *Let $(\mathbf{F}_{-\infty}, \vee, *)$ be a belt with identity and n a positive integer. For each $i = 1, 2, ..., n$, let $x_i \in \mathbf{F}_{-\infty}[x_1, ..., x_n]$ be defined by $x_i(\varepsilon_i) = 1_{\mathbf{F}_{-\infty}}$ and $x_i(u) = -\infty$ for $u \neq \varepsilon_i$.*

(i) For each integer $k \in \mathbf{N}$, $x_i^k(k\varepsilon_i) = 1_{\mathbf{F}_{-\infty}}$ and $x_i(u) = -\infty$ for $u \neq k\varepsilon_i$;

(ii) For each $(k_i, ... k_n) \in \mathbf{N^n}$, $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}(k_1\varepsilon_1 + \cdots + k_n\varepsilon_n) = 1_{\mathbf{F}_{-\infty}}$ and $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}(u) = -\infty$ for $u \neq (k_1\varepsilon_1 + \cdots + k_n\varepsilon_n)$;

(iii) $x_i^s x_j^t = x_j^t x_i^s$ for all $s, t = 1, 2, ..., n$;

(iv) $x_i^t f = f x_i^t$ for all $f \in \mathbf{F}$ and all $t \in \mathbf{N}$;

v) for every maxpolynomial g in $\mathbf{F}_{-\infty}[x_1, ..., x_n]$ there exists unique elements $a_{k_1,...,k_n} \in \mathbf{F}$, indexed by all $(k_1, ..., k_n) \in \mathbf{N}^n$ and non $-\infty$ for at most a finite number of $(k_1, ..., k_n) \in \mathbf{N}^n$, such that

$$g = \bigvee a_{k_1,...,k_n} x_1^{k_1} \ldots x_n^{k_n},$$

where the maximum is taken over all $k_1, ..., k_n \in \mathbf{N}^n$.

Proof: (i) The case for $k = 1$ is given by definition. When $k = 2$, we have

$$x_i^2(2\varepsilon_i) = \bigvee_{v+w=2\varepsilon_i} x_i(v) * x_i(w) =$$
$$x_i(\varepsilon_i) * x_i(\varepsilon_i) = 1_{\mathbf{F}}.$$

If u and v are not simultaneously ε_i, then $x_i(v) * x_i(w) = -\infty$. Assume that the formula holds for $k = n - 1$. It follows that

$$\begin{aligned} x_i^n(n\varepsilon_i) &= \bigvee_{v+w=n\varepsilon_i} x_i^{n-1}(v) * x_i(w) \\ &= x_i^{n-1}[(n-1)(\varepsilon_i)] * x_i(\varepsilon_i) \\ &= 1_{\mathbf{F}} * 1_{\mathbf{F}} \\ &= 1_{\mathbf{F}} . \end{aligned}$$

(ii)

$$\begin{aligned} & x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}(k_1\varepsilon_1 + \cdots k_n\varepsilon_n) \\ &= \bigvee_{v_1+v_2+\cdots+v_n=k_1\varepsilon_1+\cdots+k_n\varepsilon_n} x_1^{k_1}(v_1) x_2^{k_2}(v_2) \cdots x_n^{k_n}(v_n) \\ &= x_1^{k_1}(k_1\varepsilon_1) x_2^{k_2}(k_2\varepsilon_2) \cdots x_n^{k_n}(k_n\varepsilon_n) \\ &= 1_{\mathbf{F}} * 1_{\mathbf{F}} * \cdots 1_{\mathbf{F}} \\ &= 1_{\mathbf{F}} . \end{aligned}$$

If $u \neq (k_1\varepsilon_1 + \cdots + k_n\varepsilon_n)$, then it is not possible for $v_1 = k_1\varepsilon_1$, $v_2 = k_2\varepsilon_2, ..., v_n = k_n\varepsilon_n$, simultaneously. Hence, $x_1^{k_1} \cdots x_n^{k_n}(u) = -\infty$.

(iii)

$$x_i^s x_j^t(u) = 1_{\mathbf{F}} \text{ if and only if } u = s\varepsilon_i + t\varepsilon_j ,$$

but $s\varepsilon_i + t\varepsilon_j = t\varepsilon_j + s\varepsilon_i$ and

$$x_j^t x_i^s(u) = 1_{\mathbf{F}} \text{ if and only if } u = t\varepsilon_j + s\varepsilon_i .$$

Hence, $x_i^s x_j^t = x_j^t x_i^s$.

(iv)

$$\begin{aligned} x_i^t \psi_f &= \bigvee x_i^t(v)\psi_f(w) \\ &= x_i^t(t\varepsilon_i)\psi_f(u - t\varepsilon_i) \\ &= 1_{\mathbf{F}} * \psi_f \\ &= \psi_f * 1_{\mathbf{F}} \\ &= \psi_f * x_i^t(t\varepsilon_i) \\ &= \bigvee \psi_f(w) x_i^t(v) \\ &= \psi_f x_i^t . \end{aligned}$$

(v) Let $a_{k_1}, ..., a_{k_n} = g(k_1, ..., k_n)$. The $a_{k_1}, ..., a_{k_n}$ are the desired elements. To show uniqueness, if

$$\bigvee a_{k_1}, ..., a_{k_n} x_1^{k_1} \cdots x_n^{k_n} = \bigvee b_{k_1}, ..., b_{k_n} x_1^{k_1} \cdots x_m^{k_m} ,$$

then $a_j = b_j$ for $j \in \mathbf{N}^n$.

Q.E.D.

If $(\mathbf{F}_{-\infty}, \vee, *)$ is any belt, then the map $\mathbf{F}_{-\infty}[x] \to \mathbf{F}_{-\infty}[x_1, ..., x_n]$, defined by $\bigvee_{i=0}^{m} a_i * ix_1 \to \bigvee_{i=0}^{m} a_i * ix_1 * 0x_2 * \cdots * 0x_n$, is easily seen to be a monomorphism of belts. Similarly, for any subset $\{i_1, \ldots, i_k\}$ of $\{1, 2, ..., n\}$ there is a monomorphism $\mathbf{F}_{-\infty}[x] \to \mathbf{F}_{-\infty}[x_1, ..., x_n]$. The belt $\mathbf{F}_{-\infty}[x_i, ..., x_{i_k}]$ will be identified with its isomorphic image and considered to be a subbelt of $\mathbf{F}_{-\infty}[x_1, ..., x_n]$.

For the purposes of the next theorem, will shall the need the following definitions and well known theorems [19].

Definition. A category is a class $\mathcal{C}$ of objects together with

(i) a class of disjoint sets, denoted hom(A,B), one for each pair of objects in $\mathcal{C}$; (an element f' of hom(A,B) is called a morphism from A to B and is denoted $f : \mathrm{A} \to \mathrm{B}$;

(ii) for each triple (A,B,C) of objects of $\mathcal{C}$ a function

$$hom(B,C) \times hom(A,B) \to hom(A,C) ;$$

(for morphisms $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$, this function is written $(g,f) \to g \circ f$ and $g \circ f : \mathrm{A} \to \mathrm{C}$ is called the composite of f and g); all subject to the two axioms:

(I) Associativity. If $g : \mathrm{A} \to \mathrm{B}$, $h : \mathrm{B} \to \mathrm{C}$, $s : \mathrm{C} \to \mathrm{D}$ are morphisms of C, then $h \circ (g \circ s) = (h \circ g) \circ s$

(II) Identity. For each object B of C there exists a morphism $1_{\mathrm{B}} : \mathrm{B} \to \mathrm{B}$ such that for any $g : \mathrm{A} \to \mathrm{B}$, $h : \mathrm{B} \to \mathrm{C}$,

$$1_{\mathrm{B}} \circ g = g \text{ and } h \circ 1_{\mathrm{B}} = h .$$

In a category C, a morphism $g : \mathrm{A} \to \mathrm{B}$ is called an *equivalence* if there is in C a morphism $h : \mathrm{B} \to \mathrm{A}$ such that $h \circ g = 1_{\mathrm{A}}$ and $g \circ h = 1_{\mathrm{B}}$. If $g : \mathrm{A} \to \mathrm{B}$ is an equivalence, A and B are said to be equivalent.

Definition. An object I in a category $\mathcal{C}$ is said to be *universal* if for each object D of C there exist one and only one morphism $1 \to C$.

Theorem 4.2.5. *Any two universal objects in a category C are equivalent.*

Theorem 4.2.6. *Let $(\mathbf{F}_{-\infty}, \vee, *)$ and $(\mathbf{S}_{-\infty}, \vee, *)$ be commutative belts with identity and $\varphi : \mathbf{F}_{-\infty} \to \mathbf{S}_{-\infty}$ a homomorphism of belts such that $\varphi(1_{\mathbf{F}}) = \varphi(1_{\mathbf{S}})$. If $s_1, s_2, ..., s_n \in \mathbf{S}$, then there is a unique homomorphism of belts $\overline{\varphi} : \mathbf{F}_{-\infty}[x_1, ..., x_n] \to \mathbf{S}$ such that*

$\overline{\varphi} \mid \mathbf{F}_{-\infty} = \varphi$ *and* $\overline{\varphi}(x_i) = s_i$ *for* $i = 1, 2, ...n$. *This property completely determines the polynomial belt* $\mathbf{F}_{-\infty}[x_1, ...x_n]$ *up to isomorphism.*

Proof: If $g \in \mathbf{F}_{-\infty}[x_1, ..., x_n]$, then

$$g = \bigvee_{i=0}^{m} a_i x_1^{k_{i1}} \cdots x_n^{k_{in}} \quad (a_i \in \mathbf{F}_{-\infty}\,;\, k_{ij} \in \mathbf{N})$$

by Theorem 4.2.4. The map $\overline{\varphi}$ given by $\overline{\varphi}(g) = \varphi g(s_1, ..., s_n)$ is well defined map such that $\overline{\varphi} \mid \mathbf{F}_{-\infty} = \varphi$ and $\overline{\varphi}(x_i) = s_i$. We use the fact that φ is a homomorphism to show that $\overline{\varphi}$ is a homomorphism. If $g, h \in \mathbf{F}_{-\infty}[x_1, ...x_n]$, then

$$\begin{aligned}
\overline{\varphi}(g \vee h) &= \varphi(g \vee h)(s_1 \cdots s_n) \\
&= \bigvee \varphi(a_i \vee b_i) s_1 \cdots s_n \\
&= \bigvee [\varphi(a_i) \vee \varphi(b_i)] s_1 \cdots s_n \\
&= \bigvee [\varphi(a_i) s_1 \cdots s_n] \vee [\varphi(b_i) s_1 \cdots s_n] \\
&= \bigvee \varphi(a_i) s_1 \cdots s_n \quad \bigvee \varphi(b_i) s_1 \cdots s_n \\
&= \overline{\varphi}(g) \vee \overline{\varphi}(h)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\varphi}(g * h) \\
&= \overline{\varphi}\Big[\bigvee_{i_1^1=0}^{k_1^1} \bigvee_{i_1^2=0}^{k_1^2} \bigvee_{i_2^1=0}^{k_2^1} \bigvee_{i_2^2=0}^{k_2^2} \cdots \bigvee_{i_n^1=0}^{k_n^1} \bigvee_{i_n^2}^{k_n^2} a_{i_1^1 \cdots i_n^1} * b_{i_1^2 \cdots i_n^2} + \\
&\quad \big((i_1^1 + i_1^2)x_1 + \cdots (i_n^1 + i_n^2)x_n\big)\Big] \\
&= \varphi[(g * h)(s_1, ..., s_n)] \\
&= \bigvee_{i_1^1=0}^{k_1^1} \bigvee_{i_1^2=0}^{k_1^2} \bigvee_{i_2^1=0}^{k_2^1} \bigvee_{i_2^2=0}^{k_2^2} \cdots \bigvee_{i_n^1=0}^{k_n^1} \bigvee_{i_n^2}^{k_n^2} \varphi\Big(a_{i_1^1 \cdots i_n^1} * b_{i_1^2 \cdots i_n^2}\Big) + \\
&\quad \big((i_1^1 + i_1^2)s_1 + \cdots (i_n^1 + i_n^2)s_n\big)
\end{aligned}$$

$$= \bigvee_{i_1^1=0}^{k_1^1} \bigvee_{i_1^2=0}^{k_1^2} \bigvee_{i_2^1=0}^{k_2^1} \bigvee_{i_2^2=0}^{k_2^2} \cdots \bigvee_{i_n^1=0}^{k_n^1} \bigvee_{i_n^2}^{k_n^2} \varphi\left(a_{i_1^1 \cdots i_n^1}\right) * \varphi\left(b_{i_1^2 \cdots i_n^2}\right) +$$

$$\left((i_1^1 + i_1^2)s_1 + \cdots (i_n^1 + i_n^2)s_n\right)$$

$$= \left(\bigvee_{i_1^1=0}^{k_1^1} \cdots \bigvee_{i_n^1=0}^{k_n^1} \varphi\left(a_{i_1^1 \cdots i_n^1}\right) i_1^1 s_1 \cdots i_n^1 s_n \right) * \left(\bigvee_{i_1^2=0}^{k_1^2} \cdots \bigvee_{i_n^2=0}^{k_n^2} \varphi\left(b_{i_1^2 \cdots i_n^2}\right) i_1^2 s_1 \cdots i_n^2 s_n \right)$$

$$= \overline{\varphi}(f) * \overline{\varphi}(g)$$

Suppose that $\psi : \mathbf{F}_{-\infty}[x_1, ..., x_n] \to \mathbf{S}_{-\infty}$ is a homomorphism such that $\psi \mid \mathbf{F}_{-\infty} = \varphi$ and $\psi(x_i) = s_i$ for each i. Computing $\psi(g)$, we have

$$\psi(g) = \psi\left(\bigvee_{i=0}^{m} a_i x_1^{k_{i1}} \cdots x_n^{k_{in}} \right)$$

$$= \bigvee_{i=0}^{m} \psi(a_i)\psi\left(x_1^{k_{i1}}\right) \cdots \psi\left(x_n^{k_{in}}\right)$$

$$= \bigvee_{i=0}^{m} \varphi(a_i)\psi(x_1)^{k_{i1}} \cdots \psi(x_n)^{k_{in}}$$

$$= \bigvee_{i=0}^{m} \varphi(a_i) s_1^{k_{i1}} \cdots s_n^{k_{in}}$$

$$= \varphi g(s_1, ..., s_n)$$

$$= \overline{\varphi}(g) \ .$$

So, then $\psi = \overline{\varphi}$ and so $\overline{\varphi}$ is unique. Category theory is now employed to show that this property completely determines the belt $\mathbf{F}_{-\infty}[x_1, ..., x_n]$. Define a category C whose objects are all $(n+2)$-tuples, $(\psi, \mathbf{K}_{-\infty}, s_1, ...s_n)$, where $\mathbf{K}_{-\infty}$ is a commutative belt with identity, $s_i \in \mathbf{K}$ and $\psi : \mathbf{F}_{-\infty} \to \mathbf{K}_{-\infty}$ is a homomorphism with $\psi(1_{\mathbf{F}}) = 1_{\mathbf{K}}$. Our aim is to show that the object $(\iota, \mathbf{F}_{-\infty}[x_1, ..., x_n], x_1, ...x_n)$ is universal in this category.

Define a morphism in $\mathcal{C}$ from $(\psi, \mathbf{K}_{-\infty}, s_1, ...s_n)$ to $(\varphi, \mathbf{G}_{-\infty}, a_1, ..., a_n)$ as a homomorphism of belts $\rho : \mathbf{K}_{-\infty} \to \mathbf{G}_{-\infty}$ such that

$$\rho(1_{\mathbf{K}}) = 1_{\mathbf{G}}$$

$$\rho\psi = \theta$$

and

$$\rho(s) = t$$

for $i = 1, 2, ..., n$. $\rho : \mathbf{K}_{-\infty} \to \mathbf{G}_{-\infty}$ is an equivalence in $\mathcal{C}$ if and only if ρ is an isomorphism of belts. If $\iota : \mathbf{F}_{-\infty} \to \mathbf{F}_{-\infty}[x_1, ..., x_n]$ is the inclusion map, then the first part of the proof shows that $(\iota, \mathbf{F}_{-\infty}[x_1, ..., x_n], x_1, ..., x_n)$ is a universal object in C. Any other object which is universal is equivalent and so will be isomorphic. Therefore, $\mathbf{F}_{-\infty}[x_1, ..., x_n]$ is completely determined up to isomorphism by Theorem 4.2.5

Q.E.D.

Corollary 4.2.7. *Let $(\mathbf{F}_{-\infty}, \vee, *)$ be a commutative belt with identity and n a positive integer. For each k $(1 \leq d < n)$ there are isomorphisms of belts*

$$\mathbf{F}_{-\infty}[x_1, ..., x_k][x_{k+1}, ..., x_n] \widetilde{=}$$

$$\mathbf{F}_{-\infty}[x_1, ..., x_n] \widetilde{=}$$

$$\mathbf{F}_{-\infty}[x_{k+1}, ..., x_n][x_1, ..., x_k]$$

Proof: The universal mapping property established in Theorem 4.2.6 is invoked to prove the corollary. Given a homomorphism $\varphi : \mathbf{F}_{-\infty} \to \mathbf{S}_{-\infty}$ of commutative belts with identity and elements $f \in \mathbf{F}_{-\infty}[x_1, ..., x_n]$, there exists a homomorphism $\overline{\varphi} : \mathbf{F}_{-\infty}[x_1, ..., x_n] \to \mathbf{S}_{-\infty}$ such that $\overline{\varphi} \mid \mathbf{F}_{-\infty} = \varphi$ and $\overline{\varphi}(x_i) = s_i$ for $i = 1, 2, .., k$ by Theorem 4.2.6. Applying Theorem 4.2.6 with $\mathbf{F}_{-\infty}[x_1, ..., x_k]$ in place of $\mathbf{F}_{-\infty}$, yields a homomorphism $\overline{\overline{\varphi}} : \mathbf{F}_{-\infty}[x_1, .., x_k][x_{k+1}, ..., x_n] \to \mathbf{S}_{-\infty}$ such that $\overline{\overline{\varphi}} \mid \mathbf{F}_{-\infty}[x_1, ..., x_k] =$

$\overline{\varphi}$ and $\overline{\overline{\varphi}}(x_i) = s_i$ for $i = 1, 2, ..., n$. By construction $\overline{\overline{\varphi}} \mid \mathbf{F}_{-\infty} = \overline{\varphi} \mid \mathbf{F}_{-\infty} = \varphi$ and $\overline{\overline{\varphi}}(x_i) = s_i$ for $i = 1, 2, ..., n$. Suppose that $\phi : \mathbf{F}_{-\infty}[x_1, ..., x_k][x_{k+1}, ..., x_n] \rightarrow \mathbf{S}_{-\infty}$ is a homomorphism such that $\phi \mid \mathbf{F}_{-\infty} = \varphi$ and $\phi(x_i) = s_i$ for $i = 1, 2, ..., n$. The same argument used in the proof of uniqueness statement of Theorem 4.2.6 shows that $\phi \mid \mathbf{F}_{-\infty}[x_1, ..., x_k] = \overline{\varphi}$. Therefore, the uniqueness statement of Theorem 4.2.6 implies that $\phi = \overline{\overline{\varphi}}$. Consequently, $\mathbf{F}_{-\infty}[x_1, ..., x_k][x_{k+1}, ..., x_n]$ has the desired universal mapping property, whence $\mathbf{F}_{-\infty}[x_1, ..., x_k][x_{k+1}, ..., x_n] \cong \mathbf{F}_{-\infty}[x_1, ..., x_n]$ by Theorem 4.2.6. The other isomorphism is proved similarly.

Q.E.D.

CHAPTER 5
FACTORIZATION

5.1 Introduction

On the forefront of mathematical morphology research is the area of template decomposition. The area consists of taking a template with a large support and reducing it to a number of templates with smaller supports. The fundamental property which gives rise to such a study is the fact that convolutions are associative. So, if $\mathbf{t}$ is a template which has the following decomposition;

$$\mathbf{t} = \left(\boxtimes_{i=1}^{i=h} \mathbf{r}_i \right) \vee \left(\boxtimes_{j=1}^{j=k} \mathbf{s}_j \right),$$

then the convolution of an image $\mathbf{a}$, with $\mathbf{t}$, is given by

$$\mathbf{a} \boxtimes \mathbf{t} = \mathbf{a} \boxtimes \left(\boxtimes_{i=1}^{i=h} \mathbf{r}_i \ \vee \ \boxtimes_{j=1}^{j=k} \mathbf{s}_j \right)$$

$$= [(...((\mathbf{a} \boxtimes \mathbf{r_1}) \boxtimes \mathbf{r_2})...) \boxtimes \mathbf{r}_h] \vee [[(...((\mathbf{a} \boxtimes \mathbf{s}) \boxtimes \mathbf{s})...) \boxtimes \mathbf{s}]]$$

Similarly, we may use a templates decomposition to rewrite a template-template convolution.

One of the goals of any algorithm is to reduce computational complexity. Template decomposition is one of the best tools for achieving this end. A template may be represented as a maxpolynomial.

To represent a two-dimensional template as a maxpolynomial, let the coefficients a_{ij} be defined by $a_{ij} = \mathbf{t}_{(0,0)}(i,j)$ for all $(i,j) \in \mathbf{Z}^2$ [9]. Next eliminate any negative

multiples of the indeterminants from the expression

$$\bigvee_{i\in\mathbf{Z}}\bigvee_{j\in\mathbf{Z}} a_{ij}+ix+jy\,,$$

where $a_{ij}\neq-\infty$, by adding the lowest negative multiples of x and y which are present in the expression.

The adding of the indeterminants amounts to a shift of the template so that its support lies in the first quadrant. Care must be taken to keep this shift in mind when translating from maxpolynomials back to templates.

Since maxpolynomials can represent templates, factoring the maxpolynomials is one way of reducing a large template into smaller ones Maxpolynomials may be applied to the four lattice convolutions $\boxtimes$, $\boxbslash$, $\circledvee$ and $\circledwedge$.

The relationship between the additive max, $\boxtimes$, and the additive min, $\boxbslash$, is given in terms of lattice duality by

$$\mathbf{a}\boxbslash\mathbf{t}=(\mathbf{t}^*\boxtimes\mathbf{a}^*)^*,$$

where the image $\mathbf{a}^*$ is defined by $\mathbf{a}^*(\mathbf{x})=[\mathbf{a}(\mathbf{x})]^*$ and the conjugate of $\mathbf{t}\in\left(\mathbf{R}_{\pm\infty}^{\mathbf{X}}\right)^{\mathbf{Y}}$ is the template $\mathbf{t}^*\in\left(\mathbf{R}_{\pm\infty}^{\mathbf{Y}}\right)^{\mathbf{X}}$ defined by $\mathbf{t}_{\mathbf{x}}^*(\mathbf{y})=[\mathbf{t}_{\mathbf{y}}(\mathbf{x})]^*$. Similarly, there is a duality relation between the multiplicative max and the multiplicative min given by

$$\mathbf{a}\circledwedge\mathbf{t}=(\mathbf{t}^*\circledvee\mathbf{a}^*)^*$$

Here however, $\mathbf{t}\in\left(\left(\mathbf{R}_{\infty}^{\geq0}\right)^{\mathbf{X}}\right)^{\mathbf{Y}}$.

From these relations it is clear that any results obtained for $\boxtimes$ and $\circledvee$ are also results for $\boxbslash$ and $\circledwedge$.

The convolution $\circledvee$ is often computed over $(\mathbf{R}^{\geq0},\vee,\times)$. But under the map $\psi(x)=\epsilon^x$, $(\mathbf{R}_{-\infty},\vee,+)$ is isomorphic to $(\mathbf{R}^{\geq0},\vee,\times)$. Therefore, it suffices to consider only the $\boxtimes$ convolution.

Two common value sets used in the $\boxtimes$ convolution are $\mathbf{R}_{-\infty}$ and $\{-\infty, 0\}$. Section 3 is devoted to the former and Section 4 to the latter case.

5.2 Basic Properties

In this section we mention a few properties which can be applied to maxpolynomials over general belts.

Definition. If $P(x)$ is a maxpolynomial over the belt $(\mathbf{F}_{-\infty}, \vee, *)$, then $P(x)$ is a *factor* of a maxpolynomial $Q(x)$, if there exists a maxpolynomial $R(x)$ such that

$$R(x) * P(x) = Q(x).$$

The degree of a maxpolynomial is defined in the same manner as regular polynomials. That is, if $ax_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ is a monomial, then the exponent d_i is called the degree in x_i. The sum $d = d_1 + d_2 + \cdots + d_n$ is called the degree of the monomial. The ordered n–tuple $(d_1, d_2, ..., d_n)$ is the multi-degree of the monomial. The degree of a maxpolynomial is the largest degree of any of its monomial terms. There is one notable exception to these familiar rules. The degree of the $-\infty$ maxpolynomial is defined to be $-\infty$ and the degree of the zero maxpolynomials is 0. Additionally, we have the following observations about the degree of a maxpolynomial:

Theorem 5.2.1. *Let* $P, Q \in \mathbf{F}_{-\infty}[x]$, *then*

(i) $deg(Q \vee P) = \max\,(deg(Q)\,,\ deg(P))$

(ii) $deg\,(Q * P) = deg(Q) + deg(P)$

For the traditional polynomial, the way to check if $Q(x)$ divides $P(x)$ is to apply the Division Algorithm and see if there is a nonvanishing remainder. The Division Algorithm is usually stated as follows [20].

Theorem 5.2.2. *If* R *be a field and* $f, g \in \mathrm{R}[x]$, *then there exists* $q, r \in \mathbf{R}[x]$ *such that*

$$f = g * q + r \text{ and } deg(r) < deg(g).$$

The proof of this theorem relies on the group structure of R. In the case of a belt, there is not as strong a condition on $\mathbf{F}_{-\infty}$. Hence, a strict translation of the division algorithm is not possible. The next example demonstrates this shortcoming.

Example. Let $(\mathbf{F}_{-\infty}, \vee, *) = (\mathbf{R}_{-\infty}, \vee, +)$. Consider

$$f = 0 \vee (4 + x) \vee (2 + 2x)$$

and

$$g = 3 \vee (2 + x).$$

Since deg$(r) <$ deg(g), deg$(r) = 0$. So, r must be a constant. Also, deg$(f) = 2$ and since deg$(g) = 1$, it must be true that deg$(q) = 1$. Let $q = a_0 \vee (a_1 + x)$. Then

$$q + g = (a_0 + 3) \vee (a_0 + 2 + x) \vee (a_1 + 3 + x) \vee (a_1 + 2 + 2x)$$

Since r is a constant, we must have $a_1 + 2 = 2$. So, $a_1 = 0$. This implies that $a_0 + 2 = 4$. Therefore,

$$g + q = 5 \vee (4 + x) \vee (2 + 2x) \ .$$

However, there does not exist $r \in \mathbf{R}_{-\infty}$ such that $r \vee 5 = 0$.

This does not mean that there is not some analogue to the division algorithm. It is given next. Let $P(x) = a_0 \vee (a_1 + x) \vee \cdots$ and $Q(x) = b_0 \vee (b_1 + x) \vee \cdots$ be any two maxpolynomials. The reference $P(x) \geq Q(x)$ means that $a_0 \geq b_0$, $a_1 \geq b_1, \ldots$.

Theorem 5.2.3. *Let* $(\mathbf{F}_{-\infty}, \vee, *)$ *be a belt with duality such that the finite elements form a division belt and* $P, Q \in \mathbf{F}_{-\infty}[x]$. *Suppose* $deg(P) = n$ *and* $deg(Q) = m$ *with* $n \geq m$. *Let* $P(x) = a_0 \vee (a_1 + x) \vee \cdots \vee (a_n + nx)$ *and* $Q(x) = b_0 \vee (b_1 + x) \vee \cdots \vee (b_m + mx)$. *Let* K *be the set of indices such that* $b_k \neq -\infty$. *For each* $k \in \mathrm{K}$, *let*

$$h_k = \bigvee_{j=k}^{n-m+k} \left((a_j * b_k^{-1}) + (j-k)x \right) .$$

If H *is defined by*

$$H = \bigwedge_{k \in \mathrm{K}} h_k ,$$

then H *satisfies* $H(x) * Q(x) \leq P(x)$. *Furthermore, if* $R(x)$ *is any other maxpolynomial such that* $R(x) * Q(x) \leq P(x)$, *then* $H(x) \geq R(x)$.

Proof: If H_j is the j–th term of H then

$$H_j = \bigwedge_{k \in \mathrm{K}} (a_{j+k} * b_k^{-1})$$

and

$$(H * Q)_j = \bigvee_{i=0}^{j} (H_{j-i} * b_i) .$$

If there exists $k \in \mathrm{K}$ such that $k \leq j$, then

$$\begin{aligned} (H * Q)_j &= \bigvee_{k \leq j} \left(\bigwedge_{l \in \mathrm{K}} (a_{k-i+l} * b_l^{-1}) \right) * b_k \\ &\leq \bigvee_{k \leq \mathrm{K}} (a_k * b_k^{-1}) * b_k \\ &= a_k . \end{aligned}$$

Otherwise, all b_i, $i = 0, 1, ..., j$ are $-\infty$ and so, $(H * Q)_j = -\infty$.

Suppose that there exists an $R(x)$ such that $R(x) > H(x)$ and $R(x) * Q(x) \leq P(x)$. Let $R_j > H_j$. Since $m \in \mathrm{K}$, $\mathrm{K} \neq \emptyset$. Hence, there exists $k \in \mathrm{K}$ with $R_j > a_{j+k} * b_k^{-1}$.

This gives

$$(R * Q)_{j+k} = \bigvee_{i=0}^{j+k} R_{j+k-i} * b_i$$
$$> a_k * b_{k-j}^{-1} * b_{k-j} = a_k ,$$

and this is a contradiction.

Q.E.D.

Corrollary 5.2.4. *Let P, Q, and H be as in Theorem 5.2.3. Then $Q(x)$ is a factor of $P(x)$ if and only if $H(x) * Q(x) = P(x)$.*

Proof: If Q is a factor of P, then there exists $R(x)$ such that $Q(x) * R(x) = P(x)$. Therefore,

$$P(x) = R(x) * Q(x) \leq H(x) * Q(x) \leq P(x) .$$

The other direction is clear.

Q.E.D.

We define the division of two maxpolynomials as $P/Q \equiv H$.

In the example before Theorem 5.2.3, we saw how the Division Algorithm can breakdown. However, we can apply Theorem 5.2.3 to the example in a well defined way.

Example. Again, let

$$f = 0 \vee (4 + x) \vee (2 + 2x)$$

and

$$q = 3 \vee (2 + x)$$

The quotient, f/q, is calculated by first finding

$$h_1 = (0 - 3) \vee (4 - 3 + x)$$

and

$$h_2 = (4 - 2) \vee (2 - 2 + x).$$

Then,

$$f/q = h_1 \wedge h_2$$

$$= -3 \vee (0 + x).$$

Notice that $f/q + q \neq f$, which shows that q is not a factor of f.

In the next two sections extensive use is made of the fact that the finite elements of the belts under consideration form a division belt. To include the most general of possibilities, we note a procedure for when the element under consideration is $-\infty$. For the all subsequent discussions, if $x \in \mathbf{F}$, then $x - (-\infty) = x + \infty = +\infty$. However, $-\infty - (-\infty) = -\infty$.

5.3 Maxpolynomials over $(\mathbf{R}_{-\infty}, \vee, +)$

Keeping in mind the structure $(\mathbf{R}_{-\infty}, \vee, +)$, the following is noted.

Remark. A maxpolynomial $P(x)$ is a *factor* of the maxpolynomial $Q(x)$, if there exists a maxpolynomial $R(x)$ such that

$$R(x) + P(x) = Q(x) .$$

Theorem 5.3.1. *Let* $P(x) = a_0 \vee (a_1 + x) \vee \cdots \vee (a_n + nx)$, *be a maxpolynomial. If the first degree term* $(b \vee (0 + x))$ *is a factor of* P, *then* b *must satisfy*

$$a_0 - a_1 \leq b \leq a_{n-1} - a_n .$$

Proof: Let

$$P(x)/(b \vee (0 + x)) = \gamma_0 \vee (\gamma_1 + x) \vee \cdots \vee (\gamma_{n-1} + (n-1)x) .$$

By computation, if

$$P(x) = a_0 \vee (a_1 + x) \vee \cdots \vee (a_n + nx) ,$$

then it must be true that $\gamma_0 = (a_0 - b) \wedge (a_1 - 0)$. Since $(b \vee (0 + x))$ is a factor of $P(x)$, $b + \gamma_0 = a_0$. So, $\gamma_0 = a_0 - b$. Therefore, $a_0 - b \leq a_1 - 0$.

Looking at γ_{n-1}, reveals that $\gamma_{n-1} = a_n \wedge (a_{n-1} - b)$. In a similar method, it may be computed that $a_n \leq a_{n-1} - b$.

Q.E.D.

In certain cases Theorem 5.3.1 can be strengthened. Types of symmetries often have aided in the factorization of polynomials [10]. In maxpolynomials as well, these properties can be exploited. We shall need the next definition.

Definition. A maxpolynomial $P(x) = a_0 \vee (a_1 + x) \vee \cdots \vee (a_n + nx)$ is said to be skew symmetric if $a_i = -a_{n-i}$ for all $i = 0, 1, ..., n/2$. Note that this implies that if n is even then the center term is zero.

Theorem 5.3.1 can be particularly useful when dealing with a skew symmetric maxpolynomial. If it is applied to this case, the following result is obtained:

Corollary 5.3.2. *Let $P \in \mathbf{R}_{-\infty}[x]$ be skew symmetric. If the first degree term $(b \vee (0 + x))$ is a factor of P, then*

$$b = a_0 - a_1$$

It can be shown that for skew symmetric maxpolynomials of degrees 2, 3, and 4 the term $(b \vee (0 + x))$, with $b = a_0 - a_1$, is always a factor. The three cases are shown in the following results:

Let $P = a_0 \vee (0 + x) \vee (-a_0 + 2x)$. The first step is to divide P by $a_0 \vee (0 + x)$, resulting in $0 \vee (-a_0 + x)$. By adding back the term it can be seen that

$$[0 \vee (-a_0 + x)] + [a_0 \vee (0 + x)] = P\,.$$

Thus, $(b \vee (0 + x))$ is a factor in this case.

If $P = a_0 \vee (a_1 + x) \vee (-a_1 + 2x) \vee (-a_0 + 3x)$, then there are two possibilities for $P/((a_0 - a_1) \vee (0 + x))$. If $-a_1 \leq -a_0 + 2a_1$, then

$$P/((a_0 - a_1) \vee (0 + x)) = a_1 \vee (-a_1 + x) \vee (-a_0 + 2x)$$

and again

$$[a_1 \vee (-a_1 + x) \vee (-a_0 + 2x)] + [((a_0 - a_1) + (0 + x))] = P.$$

On the other hand, if $-a_1 > -a_0 + 2a_1$, then

$$P/((a_0 - a_1) \vee (0 + x)) = a_1 \vee ((-a_0 + 2a_1) + x) \vee (-a_0 + 2x)$$

and it is still true that

$$[a_1 \vee (-a_1 + x) \vee ((-a_0 + 2a_1) + 2x)] + [((a_0 - a_1) + (0 + x))] = P.$$

If $P = a_0 \vee (a_1 + x) \vee (0 + 2x) \vee (-a_1 + 3x) \vee (-a_0 + 4x)$, then there are still just two possibilities for $P/((a_0 - a_1) \vee (0 + x))$. If $-a_0 + 2a_1 \leq 0$, then

$$P/((a_0 - a_1) \vee (0 + x)) =$$

$$a_1 \vee (-a_0 + 2a_1 + x) \vee (-a_0 + a_1 + 2x) \vee (-a_0 + 3x).$$

Adding $((a_0 - a_1) \vee (0 + x))$ to this, recovers P. If $-a_0 + 2a_1 > 0$, then

$$P/((a_0 - a_1) \vee (0 + x)) = a_1 \vee (0 + x) \vee (-a_1 + 2x) \vee (-a_0 + 3x).$$

Adding back $((a_0 - a_1) \vee (0 + x))$, again gives us P.

Of course it is not always true that $(b \vee (0 + x))$ is a factor. A counter example is of degree 5. If

$$Q(x) = 1 \vee (-2 + x) \vee (-1 + 2x) \vee (1 + 3x)$$

$$\vee(2 + 4x) \vee (-1 + 5x)\ ,$$

then

$$Q(x)/(3 \vee (0 + x)) = -2 \vee (-5 + x) \vee (-4 + 2x)$$

$$\vee(-2 + 3x) \vee (-1 + 4x)\ .$$

Now, by adding back $(3 \vee (0 + x))$, we see that Q is not recovered.

Theorem 5.3.3. *Let* $P(x) = a_0 \vee (a_1 + x) \vee \ldots \vee (a_n + nx)$ *be a maxpolynomial with* $a_i \neq -\infty$ *for* $i = 0, 1, ..., n$. *Compute the numbers* $b_1 = a_0 - a_1, b_2 = a_1 - a_2, \ldots, b_n = a_{n-1} - a_n$. *If there exists a number* j *such that*

$$\max_{i=1,j} b_i \leq \min_{i=j+1,n} b_i,$$

then $P(x)$ *can be factored into a maxpolynomial of degree* j *and a maxpolynomial of degree* $n - j$.

Proof: Define

$$P_0 = a_0 \vee (a_1 + x) \vee \cdots \vee (a_j + jx)$$

and

$$P_1 = 0 \vee (a_{j+1} - a_j + x) \vee (a_{j+2} - a_j + 2x) \vee \cdots \vee (a_n - a_j + (n-j)x).$$

Let $P_0 + P_1 = c_0 \vee (c_1 + x) \vee \cdots \vee (c_n + nx)$. If $k \leq j$, then for $i = 0, 1, ..., k-1$

$$a_{j+l} - a_{j+l+1} \geq a_{i+l} - a_{i+l+1}\ ,\ l = 0, \ldots, k-i-1.$$

Hence,

$$\sum_{l=0}^{k-i-1} (a_{j+l} - a_{j+l+1}) \geq \sum_{l=0}^{k-i-1} (a_{i+l} - a_{i+l+1}).$$

This gives

$$\begin{aligned} a_k &= a_k \vee \left(\bigvee_{i=0}^{k-1} (a_i + a_{j+k-i} - a_j) \right) \\ &= \bigvee_{i=0}^{k} (a_i + a_{j+k-i} - a_j) \\ &= c_k. \end{aligned}$$

If $k \geq j$, then for $i = 0, 1, ..., j-1$

$$a_{k+l} - a_{k+l+1} \geq a_{i+l} - a_{i+l+1}\ ,\ l = 0, ..., j-i-1.$$

So,

$$\sum_{l=0}^{j-i-1} (a_{k+l} - a_{k+l+1}) \geq \sum_{l=0}^{j-i-1} (a_{i+l} - a_{i+l+1}),$$

which gives

$$\begin{aligned} a_k &= a_k \vee \left(\bigvee_{i=0}^{k} (a_i + a_{j+k-i} - a_j) \right) \\ &= \bigvee_{i=0}^{k} (a_i + a_{j+k-i} - a_j) \\ &= c_k. \end{aligned}$$

Q.E.D.

This theorem can be applied to some cases in which some of the coefficients are $-\infty$. The next corollary shows that a strict inequality on the differences of the coefficients is all that is needed.

Corollary 5.3.4. *Let* $P(x) = a_0 \vee (a_1 + x) \vee \ldots \vee (a_n + nx)$ *be a maxpolynomial. Compute the numbers* $b_1 = a_0 - a_1, b_2 = a_1 - a_2, \ldots, b_n = a_{n-1} - a_n$. *If there exists a number* j *such that*

$$\max_{i=1,j} b_i < \min_{i=j+1,n} b_i,$$

then $P(x)$ *can be factored into a maxpolynomial of degree* j *and a maxpolynomial of degree* $n - j$.

Proof: The strict inequality means that $a_j \neq -\infty$. The proof is the same as that of the previous theorem.

Q.E.D.

Example. If

$$P = (0 + x) \vee (2 + 2x) \vee (0 + 3x),$$

then

$$b_1 = -\infty - 0 = -\infty,$$

$$b_2 = 0 - 2 = -2,$$

$$b_3 = 2 - 0 = 2.$$

Corollary 5.3.4 says that one possible factorization is

$$P = [(0 + x) \vee (2 + 2x)] + [0 \vee (-2 + x)].$$

Example. This example shows that the conditions of Theorem 5.3.3 are only sufficient conditions. Let

$$P = 5 \vee (3 + x) \vee (5 + 2x) \vee (4 + 3x) \vee (4 + 4x) \vee (4 + 5x).$$

This maxpolynomial may be factored as

$$P = [2 \vee (0 + x) \vee (2 + 2x)]$$

$$+[3 \vee (1 + x) \vee (2 + 2x) \vee (2 + 3x)].$$

However, it does not meet the conditions of Theorem 5.3.3.

One class of maxpolynomials which is common in template representation is symmetric maxpolynomials. Symmetric polynomials were studied by Manseur [10]. We follow that definition for symmetric polynomials.

Definition. A maxpolynomial $P(x) = a_0 \vee (a_1 + x) \vee \cdots \vee (a_n + nx)$ is *symmetric with respect to n*, if $a_i = a_{n-i}$ for all $i = 1, 2, ..., n$.

When a maxpolynomial is said to be symmetric, we shall always mean with respect to the degree of the maxpolynomial.

Corollary 5.3.5. *If P is a symmetric maxpolynomial of even degree such that the coefficients increase from a_0 to $a_{n/2}$, then P factors into two maxpolynomials of degree $n/2$.*

Proof: The conditions on P imply that the numbers b_i are greater than or equal to 0 for $i = 1, 2, ... n/2$ and less than or equal to 0 for $i = \frac{n}{2} + 1, ..., n$. Hence, Theorem 5.3.3 applies.

Q.E.D.

When the conditions of the corollary are met and $a_{\frac{n}{2}}$ is even, $a_{\frac{n}{2}}/2$ may be subtracted from P_0 and added to P_1. Doing so results in a factorization which shall be shown to be valuable in the decomposition of two variable maxpolynomials. This corollary will be used in Theorem 5.3.14.

Q.E.D.

Theorem 5.3.6. *If $P(x)$ is a symmetric maxpolynomial of even degree and P factors into first degree terms, then all the factors appear in conjugate pairs.*

Proof: Let $P_2(x) = 0 \vee (a_1 + x) \vee (0 + 2x)$. Since P_2 factors, the factors must have constant terms which add to give the constant term of P_2 and the coefficients of the highest terms must add to give the highest term. Therefore, if $(c_0 \vee (c_1 + x))$ is a factor then the other factor must be $(-c_0 \vee (-c_1 + x))$.

Next, assume that the results holds for a maxpolynomial of degree n.

Given P_{n+2}, the reducibility criterion provides that

$$P_{n+2} = P_n + (b_0 \vee (b_1 + x)) + \left(b_0' \vee \left(b_1' + x\right)\right).$$

The constant term of P_n is 0. Therefore, $P_n + (b_0 \vee (b_1 + x))$ has b_0 as the constant term. Also P_{n+2} has a constant term of 0.

Hence, b_0' must equal $-b_0$. Similarly, it is shown that $b_1' = -b_1$.

Q.E.D.

Theorem 5.3.7. *Let* $P = 0 \vee (a_1 + x) \vee \cdots \vee (a_1 + (n-1)x) \vee (0 + nx)$ *be a symmetric maxpolynomial of even degree. If* $(b \vee (0 + x))$ *is a factor of* P, *then* $b \leq a_1$.

Proof: Suppose that $b > a_1$ and $(b \vee (0 + x))$ is a factor of P. The division theorem is used to calculate $P/(b \vee (0 + x))$. The candidates for the coefficient of $(n-1)x$ are $a_1 - b$ and 0. In order for

$$[P/(b \vee (0 + x))] + (b \vee (0 + x)) = P,$$

it must be true that $a_1 - b \geq 0$. Thus, there is a contradiction.

Theorem 5.3.8. *Let* $P = 0 \vee (a_1 + x) \vee (a_2 + 2x) \vee \cdots \vee (a_1 + (n-1)x) \vee (0 + nx)$ *be a symmetric maxpolynomial of even degree. Define* $c_1 = a_1$ *and* $c_i = a_i - a_{i-1}$ *for* $i = 2, 3, ..., n/2$. *The maxpolynomial* P *factors into first degree terms if and only if*

$$\begin{aligned} P = {} & (c_1 \vee (0 + x)) + (-c_1 \vee (0 + x)) \\ & + (c_2 \vee (0 + x)) + (-c_2 \vee (0 + x)) \\ & + \cdots \\ & + \left(c_{n/2} \vee (0 + x)\right) + \left(-c_{n/2} \vee (0 + x)\right). \end{aligned}$$

Proof: Suppose that P factors into first degree terms. By theorem 5.3.6

$$P_n = (d_1 \vee (0+x)) + (-d_1 \vee (0+x)) +$$

$$\cdots + \left(d_{n/2} \vee (0+x)\right) + \left(-d_{n/2} \vee (0+x)\right).$$

An ordering on the d_i, such that $d_1 \geq d_2 \geq \cdots \geq d_{\frac{n}{2}}$, may be assumed. Combining conjugates first, yields

$$P = (0 \vee (d_1 + x) \vee (0+2x))$$

$$+((0 \vee (d_2 + x) \vee (0+2x)))$$

$$\vdots$$

$$+\left(\left(0 \vee \left(d_{n/2} + x\right) \vee (0+2x)\right)\right).$$

Using the ordering on the d_i, we begin combining more terms. The first step yields

$$P = (0 \vee (d_1 + x) \vee (d_1 + d_2 + 2x) \vee (d_1 + 3x) \vee (0+4x))$$

$$+(0 \vee (d_3 + x) \vee (0+2x)) + (0 \vee (d_4 + x) \vee 0 + 2x) +$$

$$\cdots + \left(0 \vee \left(d_{n/2} + x\right) \vee (0+2x)\right).$$

Continuing in this way results in

$$P = 0 \vee (d_1 + x) \vee (d_1 + d_2 + 2x) \vee (d_1 + d_2 + d_3 + 3x)$$

$$\vee \cdots \vee (d_1 + d_2 + \cdots + d_{n/2} + (n/2)x) \vee \cdots$$

$$\vee (d_1 + (n-1)x) \vee (0 + nx).$$

Thus, $d_1 = a_1$ and $d_i = a_i - a_{i-1}$.

Q.E.D.

Example. Consider the template

$$\mathbf{p} = \quad 0 \quad 3\lambda \quad 4\lambda \quad 3\lambda \quad 0 \quad .$$

where λ is a free parameter. The corresponding maxpolynomial is

$$0 \vee (3\lambda + x) \vee (4\lambda + 2x) \vee (3\lambda + 3x) \vee (0 + 4x).$$

According to Theorem 5.3.8, this factors as

$$(-\lambda \vee (0 + x)) + (\lambda \vee (0 + x))$$

$$+ (-3\lambda \vee (0 + x)) + (3\lambda \vee (0 + x)).$$

The corresponding templates are

$$\mathbf{p} = \begin{matrix} -\lambda & 0 \end{matrix} \qquad \begin{matrix} \lambda & 0 \end{matrix} \qquad \begin{matrix} -3\lambda & 0 \end{matrix} \qquad \begin{matrix} 3\lambda & 0 \end{matrix} \quad .$$

Theorem 5.3.8 leads immediately to several observations. One is that a symmetric maxpolynomial can only factor completely if all the terms are positive. Another observation is shown in the next theorem.

Corollary 5.3.9. *If $P = a_0 \vee (a_1 + x) \vee \cdots \vee (a_0 + nx)$ is symmetric and factors into first degree terms, then $a_k \leq a_{k+1}$ for $k = 0, 1, ..., \frac{n}{2}$.*

Corollary 5.3.10. *If $P = a_0 \vee (a_1 + x) \vee \cdots \vee (a_0 + nx)$ is symmetric and factors into first degree terms, then $a_{i+1} - a_i \leq a_{i+2} - a_{i+1}$ for $i = 0, 1, ..., \frac{n}{2} - 2$.*

Theorem 5.3.11. *If P is symmetric of odd degree and the coefficients increase from a_1 to $a_{\frac{n}{2}}$, then there exists Q, symmetric of even degree, such that*

$$Q + (0 \vee (0 + x)) = P.$$

Proof: Let

$$P = 0 \vee (a_1 + x) \vee \cdots \vee \left(a_{\frac{n-1}{2}} + \left(\frac{n-1}{2}\right)x\right)$$
$$\vee \left(a_{\frac{n-1}{2}} + \left(\frac{n-1}{2} + 1\right)x\right) \vee \cdots \vee (a_1 + (n-1)x) \vee (0 + nx).$$

Next, divide P by $(0 \vee (0 + x))$. Recall that $P/(0 \vee (0 + x)) = \bigwedge_{j=1}^{2} h_j$, where the coefficients of h_1 are

$$\left(0, a_1, a_2, ..., a_{\frac{n-1}{2}}, a_{\frac{n-1}{2}}, .., a_2, a_1\right)$$

and the coefficients of h_2 are

$$\left(a_1, a_2, ..., a_{\frac{n-1}{2}}, a_{\frac{n-1}{2}}, ..., a_2, a_1, 0\right).$$

Thus, the coefficients of $P/(0 \vee (0+x))$ are

$$\left(0, a_1, a_2, ..., a_{\frac{n-1}{2}}, ..., a_2, a_1, 0\right).$$

By calculation $P/(0 \vee (0+x)) + (0 \vee (0+x)) = P$.

Q.E.D.

We now begin the consideration of two variable maxpolynomials. One of the most desirable factorizations of two variable maxpolynomials is a decomposition into two one variable maxpolynomials. First, this special case. Note that the next theorem is an extension of the result for templates given by Li [21].

Theorem 5.3.12. *Let* $T(x,y) = \bigvee_{i=0}^{m} \bigvee_{j=0}^{n} (t_{ij} + ix + jy)$ *be a maxpolynomial in two variables with* $t_{mn} \neq -\infty$, *then* $T(x,y) = P(x) + Q(y)$ *if and only if* $t_{ij} = t_{in} + t_{mj} - t_{mn}$ *for* $0 \leq i \leq m$ *and* $0 \leq j \leq n$.

Proof: Suppose that $T = P + Q$. Let

$$P = a_0 \vee (a_1 + x) \vee \cdots \vee (a_m + mx)$$

and

$$Q = b_0 \vee (b_1 + y) \vee \cdots \vee (b_n + ny),$$

where $a_m \neq -\infty$ and $b_n \neq -\infty$. It may assumed that $a_n = 0$ and thus that $t_{mj} = b_j$ for $j = 0, 1, ..., n$. In particular, note that $t_{mn} \neq -\infty$. The relation $t_{ij} + ix + jy = (a_i + ix) + (b_j + jy)$ also holds. However, b_j may be calculated by $b_j = t_{mj}$ and $a_i = t_{im} - t_{mn}$.

If T satisfies $t_{ij} = t_{in} + t_{mj} - t_{mn}$ for $0 \leq i \leq n$, $0 \leq j \leq m$, then define

$$P = \bigvee_{j=0}^{n} (t_{mj} - t_{mn} + jx)$$

and

$$Q = \bigvee_{i=0}^{m} (t_{in} + ix) .$$

Calculation shows that $P + Q = T$.

Q.E.D.

Maxpolynomials, or corresponding templates, which satisfy the conditions of this theorem are referred to as separable.

Example. A parabolic structuring element can be used to bring out texture information and suppress both point noise and white noise [22]. In the following parabolic template, t, the parameter λ is a free parameter.

Let

$$\mathbf{t} = \begin{array}{ccccc} 0 & 3\lambda & 4\lambda & 3\lambda & 0 \\ 3\lambda & 6\lambda & 7\lambda & 6\lambda & 3\lambda \\ 4\lambda & 7\lambda & 8\lambda & 7\lambda & 4\lambda \\ 3\lambda & 6\lambda & 7\lambda & 6\lambda & 3\lambda \\ 0 & 3\lambda & 4\lambda & 3\lambda & 0 \end{array} .$$

According to Theorem 5.3.12 this template is separable. Hence, it may be decomposed into a row template and a column template. So $\mathbf{t}=\mathbf{p} \boxtimes \mathbf{q}$, where

$$\mathbf{p} = \begin{array}{ccccc} 0 & 3\lambda & 4\lambda & 3\lambda & 0 \end{array} \quad \text{and} \quad \mathbf{q} = \begin{array}{c} 0 \\ 3\lambda \\ 4\lambda \\ 3\lambda \\ 0 \end{array} .$$

Recall that a rectangular template is one whose support is a subset of a rectangle.

The previous results on the separability of templates was limited to templates whose support was identical to the smallest rectangle containing the support [21]. Theorem 5.3.12 applies to a wider class of templates. Consider the following template, t.

Example. Let

$$\mathbf{t} = \begin{array}{ccc} 0 & & 0 \\ & & \\ 0 & & 0 \end{array} .$$

The corresponding maxpolynomial is given by

$$0 \vee (0 + 2x) \vee (0 + 2y) \vee (0 + 2x + 2y) .$$

This factors as

$$[0 \vee (0 + 2x)] + [0 \vee (0 + 2y)]$$

or

0 0 0 .

0

There are often cases when a two variable maxpolynomial is not separable. In such cases, it may be possible to apply the one variable theorems already presented to reduce the two variable maxpolynomial.

For the next definition, let t be is a translation invariant rectangular template with maxpolynomial representation $T(x,y) = \bigvee_{i=0}^{m} \bigvee_{j=0}^{n} (t_{ij} + ix + jy)$.

Definition. The *boundary maxpolynomials of a rectangular translation invariant template* are the maxpolynomials $P_1 = \bigvee_{i=0}^{m} (t_{i0} + ix)$, $P_2 = \bigvee_{j=0}^{n} (t_{0j} + jy)$, $P_3 = \bigvee_{i=0}^{m} (t_{in} + ix + ny)$, and $P_4 = \bigvee_{j=0}^{n} (t_{mj} + mx + jy)$.

If t is a rectangular template, then the boundary maxpolynomials may be obtained by first finding the maxpolynomial that corresponds to t and then isolating certain coefficients. The coefficients to isolate are from the terms which have the highest degree in each variable and the lowest degree in each variable. This will give the four boundary maxpolynomials.

Example. Let

$$\mathtt{t} = \begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & \\ 0 & 1 & 2 & 1 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array}.$$

The boundary maxpolynomial for this template are

$$P_1 = 0 + 2x$$

$$P_2 = 0 + 2y$$

$$P_3 = 0 + 2x + 4y$$

$$P_4 = 0 + 4x + 2y$$

Suppose s and t are two rectangular templates. To compute the boundary maxpolynomial of their convolution, it is only necessary to add corresponding boundary maxpolynomials from the two templates. This is obvious when one considers that, for example, the terms with lowest degree in x from $\mathtt{s} \boxtimes \mathtt{t}$ are obtained by adding the terms with the lowest degree in x from s with those of t.

These observations are recorded in the next proposition.

Proposition 5.3.13. *Suppose that* t *is a rectangular template and* $A^1(x,y)$, $A^2(x,y), A^3(x,y)$, *and* $A^4(x,y)$ *correspond to a counterclockwise representation of the boundary of* t *where any* $A^i(x,y)$ *could be a monomial. If* t

is reducible into the convolution of two rectangular templates, then there exists factorizations of $A^1(x,y),...,A^4(x,y)$,

$$A^1 = A_1^1 + A_2^1$$

$$\vdots$$

$$A_1^4 = A_1^4 + A_2^4$$

such that $A_1^1(x,y)$, $A_1^2(x,y)$, $A_1^3(x,y)$, $A_1^4(x,y)$ *and* $A_2^1(x,y)$, $A_2^2(x,y)$, $A_2^3(x,y)$, $A_2^4(x,y)$ *correspond to a counterclockwise representation of the boundary of two templates.*

Proof: Suppose that t=s$\boxtimes$r. Let $A_1^1(x,y),...,A_1^4(x,y)$ correspond to the boundary of s and $A_2^1(x,y),...,A_2^4(x,y)$ correspond to the boundary of r.

Q.E.D.

Definition. A maxpolynomial in two variables $P(x,y) = \bigvee_{i=0}^{m}\bigvee_{j=0}^{n}(t_{ij}+ix+jy) = \bigvee_{i=0}^{m}(ix+P_i(y))$ is *symmetric with respect to y*, if each $P_i(y)$ is symmetric with respect to n.

A similar definition can be given for the variable x.

Definition. A maxpolynomial in two variables is *symmetric*, if it is symmetric with respect to both x and y.

Theorem 5.3.14. *Suppose* $P(x,y) = \bigvee_{i=0}^{m}\bigvee_{j=0}^{n}(a_{ij}+ix+jy)$ *corresponds to a rectangular template, and* T *is symmetric with both* m *and* n *even. If*

$$a_{00} \leq a_{10} \leq \cdots \leq a_{\frac{m}{2}0},$$

$$a_{00} \le a_{01} \le \cdots \le a_{0\frac{n}{2}} ,$$

$a_{\frac{m}{2}0} = a_{0\frac{n}{2}}$, $a_{0\frac{n}{2}}$ *is even* $a_{ij} \ge a_{0\frac{n}{2}}$ *for* $1 \le i \le m-1$ *and* $1 \le j \le n-1$, *then there exists maxpolynomials* $P(x,y), Q(x,y)$, *and* $R(x,y)$ *such that*

$$T(x,y) = [P(x,y) + Q(x,y)] \vee R(x,y)$$

where

$$R(x,y) = \bigvee_{j=1}^{n-1} \bigvee_{i=1}^{m-1} a_{ij} + ix + jy .$$

Proof: Since the support of the template may not be rectangular, several of the coefficients of T may be $-\infty$. The boundary maxpolynomials are symmetric with a center term that is even. Even with certain coefficients equal to $-\infty$, Corollary 5.3.5 and the procedure in the comments that follow it, may be applied to each of the boundary maxpolynomials. The results for each of the boundary maxpolynomials are

$$\begin{aligned}
&\bigvee_{i=0}^{m} a_{i0} + ix \\
&= \left(a_{00} - \frac{1}{2}a_{\frac{m}{2}0}\right) \vee \left(a_{10} - \frac{1}{2}a_{\frac{m}{2}0} + x\right) \vee \cdots \vee \left(\frac{1}{2}a_{\frac{m}{2}0} + \frac{m}{2}x\right) \\
&+ \left(\frac{1}{2}a_{\frac{m}{2}0}\right) \vee \left(a_{(\frac{m}{2}-1)0} - \frac{1}{2}\left(a_{\frac{m}{2}0}\right) + x\right) \vee \cdots \vee \left(a_{00} - \frac{1}{2}a_{\frac{m}{2}0} + \frac{m}{2}x\right) \\
&= A_1 + A_2 ,
\end{aligned}$$

$$\begin{aligned}
&\bigvee_{i=0}^{m} a_{in} + ix + ny \\
&= \frac{n}{2}y + \left[\left(a_{0n} - \frac{1}{2}a_{\frac{m}{2}n}\right) \vee \left(a_{1n} - \frac{1}{2}a_{\frac{m}{2}n} + x\right) \vee \cdots \vee \left(\frac{1}{2}a_{\frac{m}{2}n} + \left(\frac{m}{2}\right)x\right)\right] \\
&+ \frac{n}{2}y + \left[\left(\frac{1}{2}a_{\frac{m}{2}n}\right) \vee \left(a_{(\frac{m}{2}-1)n} - \frac{1}{2}\left(a_{\frac{m}{2}n}\right) + x\right) \vee \cdots \vee \left(a_{0n} - \frac{1}{2}a_{\frac{m}{2}n} + \frac{m}{2}x\right)\right] \\
&= B_1 + B_2 ,
\end{aligned}$$

$$\bigvee_{i=0}^{n} a_{0i} + iy$$

$$= \left(a_{00} - \frac{1}{2}a_{0\frac{n}{2}}\right) \vee \left(a_{01} - \frac{1}{2}a_{0\frac{n}{2}} + y\right) \vee \cdots \vee \left(\frac{1}{2}a_{0\frac{n}{2}} + \frac{n}{2}y\right)$$

$$+ \left(\frac{1}{2}a_{0\frac{n}{2}}\right) \vee \left(a_{0(\frac{n}{2}-1)} - \frac{1}{2}\left(a_{0\frac{n}{2}}\right) + y\right) \vee \cdots \vee \left(a_{00} - \frac{1}{2}a_{0\frac{n}{2}} + \frac{n}{2}y\right)$$

$$= C_1 + C_2 .$$

and

$$\bigvee_{i=0}^{n} a_{mi} + mx + iy$$

$$= \frac{m}{2}x + \left[\left(a_{m0} - \frac{1}{2}a_{m\frac{n}{2}}\right) \vee \left(a_{m1} - \frac{1}{2}a_{m\frac{n}{2}} + x\right) \vee \cdots \vee \left(\frac{1}{2}a_{m\frac{n}{2}} + \left(\frac{n}{2}\right)y\right)\right]$$

$$+ \frac{m}{2}x + \left[\left(\frac{1}{2}a_{m\frac{n}{2}}\right) \vee \left(a_{m(\frac{n}{2}-1)} - \frac{1}{2}\left(a_{m\frac{n}{2}}\right) + y\right) \vee \cdots \vee \left(a_{m0} - \frac{1}{2}a_{m\frac{n}{2}} + \frac{n}{2}y\right)\right]$$

$$= D_1 + D_2 .$$

We now use $a_{\frac{m}{2}0} = a_{0\frac{n}{2}}$, and define

$$P(x, y) = A_1 \vee D_2 \vee B_2 \vee C_1$$

and

$$Q(x. y) = A_2 \vee D_1 \vee B_1 \vee C_2 .$$

Then

$$P + Q = (A_1 + A_2) \vee [A_2 + (D_2 \vee B_2 \vee C_1)]$$

$$+ (D_1 + D_2) \vee [D_1 + (A_1 \vee B_2 \vee C_1)]$$

$$+ (B_1 + B_2) \vee [B_1 + (A_1 \vee D_2 \vee C_1)]$$

$$+ (C_1 + C_2) \vee [C_2 + (A_1 \vee D_2 \vee B_2)] .$$

Thus, $P + Q$ gives back the boundary maxpolynomials of T. The terms from

$$[A_2 + (D_2 \vee B_2 \vee C_1)]$$

$$[D_1 + (A_1 \vee B_2 \vee C_1)]$$

$$[B_1 + (A_1 \vee D_2 \vee C_1)]$$

$$[C_2 + (A_1 \vee D_2 \vee B_2)]$$

form the interior of $P+Q$. The largest terms of P added to the largest terms of Q naturally give the largest terms of $P + Q$. Those terms from P are $\frac{1}{2}a_{0\frac{n}{2}}$ and $\left(\frac{1}{2}a_{0\frac{n}{2}} + \frac{n}{2}x + \frac{n}{2}y\right)$ and those from Q are $\left(\frac{1}{2}a_{0\frac{n}{2}} + \frac{n}{2}x\right)$ and $\left(\frac{1}{2}a_{0\frac{n}{2}} + \frac{n}{2}y\right)$. Notice that in $P + Q$, these terms will be in the boundary maxpolynomials. Hence, the condition $a_{ij} \geq a_{0\frac{n}{2}}$ for $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, insures that the coefficients produced by $P + Q$ are not larger than the coefficients of T.

Thus, it is possible to define

$$R(x,y) = \bigvee_{j=1}^{n-1} \bigvee_{i=1}^{m-1} a_{ij} + ix + jy\,.$$

Q.E.D.

Example. This example demonstrates the use of Theorem 5.3.14. The following template is used for location determination [23]. Let

$$\mathbf{t} = \begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & \\ 0 & 1 & 2 & 1 & 0 \\ & 0 & 1 & 0 & \\ & & 0 & & \end{array}\,.$$

The maxpolynomial which corresponds to this template is

$$T = (0 + 2y) \vee (0 + x + y) \vee (1 + x + 2y) \vee (0 + x + 3y)$$

$$\vee (0 + 2x) \vee (1 + 2x + y) \vee (2 + 2x + 2y) \vee (1 + 2x + 3y) \vee (0 + 2x + 4y)$$

$$\vee (0 + 3x + y) \vee (1 + 3x + 2y) \vee (0 + 3x + 3y) \vee (0 + 4x + 2y)\,.$$

This factors according to theorem 5.3.14. The result is

$$T = [0 \vee (0 + 2x + 2y) \ + \ (0 + 2y) \vee (0 + 2x)]$$

$$\vee [0 \vee (1 + y) \vee (0 + 2y)$$

$$\vee (1 + x) \vee (2 + x + y) \vee (1 + x + 2y)$$

$$\vee (0 + 2x) \vee (1 + 2x + y) \vee (0 + 2x + 2y)]\,.$$

Thus, we have $T = [P + Q] \vee R$, where

$$P = 0 \vee (0 + 2x + 2y),$$

$$Q = \ (0 + 2y) \vee (0 + 2x)$$

and

$$R = [0 \vee (1 + y) \vee (0 + 2y)$$

$$\vee (1 + x) \vee (2 + x + y) \vee (1 + x + 2y)$$

$$\vee (0 + 2x) \vee (1 + 2x + y) \vee (0 + 2x + 2y)]\,.$$

The template representation is $\mathbf{t} = \mathbf{p} \boxtimes \mathbf{q} \vee \mathbf{r}$, where

$$\mathbf{p} = \begin{matrix} 0 & \\ & 0 \end{matrix} \qquad \mathbf{q} = \begin{matrix} & 0 \\ 0 & \end{matrix}$$

and

$$\mathbf{r} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 2 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} .$$

Theorem 5.3.15. *Suppose that* $T(x,y) = \bigvee_{j=0}^{n} \bigvee_{i=0}^{m} a_{ij} + ix + jy$ *is a symmetric maxpolynomial such that the boundary maxpolynomials factor into first degree terms with* $m, n \geq 4$. *If*

$$a_{ij} \geq (a_{0\frac{n}{2}} - a_{01} + a_{10}) \vee \left(a_{\frac{m}{2}0} - a_{10} + a_{01}\right)$$

for $1 \leq j \leq n-1$ *and* $1 \leq i \leq m-1$, *then there exists maxpolynomials* $P(x,y), Q(x,y)$, *and* $R(x,y)$ *such that*

$$T(x,y) = [P(x,y) + Q(x,y)] \vee R(x,y)\,,$$

where

$$R(x,y) = \bigvee_{j=1}^{n-1} \bigvee_{i=1}^{m-1} a_{ij} + ix + jy\,.$$

Proof: If

$$\bigvee_{i=0}^{m} a_{i0} + ix = A$$

$$\bigvee_{i=0}^{m} a_{in} + ix + ny = B$$

$$\bigvee_{j=0}^{n} a_{0j} + jy = C$$

$$\bigvee_{j=0}^{n} a_{ni} + nx + jy = D\,,$$

then A can be written as

$$(0 \vee (a_{01} + x) \vee (0 + 2x)) + A_2 \,,$$

B as

$$((0 + 2y) \vee (a_{01} + x + 2y) \vee (0 + 2x + 2y)) + B_2 \,,$$

C as

$$(0 \vee (a_{10} + y) \vee (0 + 2y)) + C_2 \,,$$

and D as

$$((0 + 2x) \vee (a_{01} + 2x + y) \vee (0 + 2x + 2y)) + D_2 \,.$$

Define

$$\begin{aligned} P(x,y) = {} & (0 \vee (a_{01} + x) \vee (0 + 2x)) \\ & \vee ((0 + 2x) \vee (a_{1m} + 2x + y) \vee (0 + 2x + 2y)) \\ & \vee ((0 + 2y) \vee (a_{1n} + x + 2y) \vee (0 + 2x + 2y)) \\ & \vee (0 \vee (a_{01} + y) \vee (0 + 2y)) \end{aligned}$$

and

$$Q(x,y) = A_2 \vee B_2 \vee C_2 \vee D_2 \,.$$

The proof proceeds as before, noting that the highest term of $P + Q$ is the maximum of the largest terms of P added to the largest of Q. This is given by

$$(a_{0\frac{n}{2}} - a_{01} + a_{10}) \vee \left(a_{\frac{m}{2}0} - a_{10} + a_{01}\right) \,.$$

Q.E.D.

Example. To demonstrate Theorem 5.3.15, we again look at a template which is used for location determination [23]. Let

$$\mathbf{t} = \begin{array}{ccccc} -2 & -2 & -2 & -2 & -2 \\ -2 & -1 & -1 & -1 & -2 \\ -2 & -1 & 0 & -1 & -2 \\ -2 & -1 & -1 & -1 & -2 \\ -2 & -2 & -2 & -2 & -2 \end{array} .$$

The template decomposition is given by $\mathbf{t} = \mathbf{p} \boxtimes \mathbf{q} \vee \mathbf{r}$, where

$$\mathbf{p} = \begin{array}{ccc} -2 & -2 & -2 \\ -2 & & -2 \\ -2 & -2 & -2 \end{array} \qquad \mathbf{q} = \begin{array}{ccc} -2 & -2 & -2 \\ -2 & & -2 \\ -2 & -2 & -2 \end{array}$$

$$\text{and} \quad \mathbf{r} = \begin{array}{ccc} -1 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & -1 \end{array} .$$

Factorization methods for polynomials are often recursive. If a symmetric polynomial is factored as $T = P * Q + R$, then R is symmetric and can usually be factored by the same theorem which led to the factorization of T [10, Corollary 2 to Theorem 3.1]. However, the same is not true for maxpolynomials. As is demonstrated in the next example, there

may exist a factorization $T = (P + Q) \vee R$, but R does not satisfy either the hypotheses of Theorem 5.3.14 or Theorem 5.3.15.

Example. Let

$$\mathbf{t} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & 2 & 1 \\ \hline 2 & 4 & 5 & 7 & 5 & 4 & 2 \\ \hline 3 & 5 & 5 & 4 & 5 & 5 & 3 \\ \hline 3 & 7 & 4 & 3 & 4 & 7 & 3 \\ \hline 3 & 5 & 5 & 4 & 5 & 5 & 3 \\ \hline 2 & 4 & 5 & 7 & 5 & 4 & 2 \\ \hline 1 & 2 & 3 & 3 & 3 & 2 & 1 \\ \hline \end{array} .$$

The template $\mathbf{t}$ may be decomposed by either theorem 5.3.14 or Theorem 5.3.15. However, in both cases we will have

$$\mathbf{r} = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & 7 & 5 & 4 \\ \hline 5 & 5 & 4 & 5 & 5 \\ \hline 7 & 4 & 3 & 4 & 7 \\ \hline 5 & 5 & 4 & 5 & 5 \\ \hline 4 & 5 & 7 & 5 & 4 \\ \hline \end{array} .$$

The template $\mathbf{r}$ does not satisfy the hypothesis of either theorem.

To show that $\mathbf{r}$ can not be decomposed into symmetric templates, suppose that such templates exist. Let $\mathbf{r} = \mathbf{s}_1 \boxtimes \mathbf{s}_2 \vee \mathbf{r}_2$, where

$$\mathbf{s}_1 = \begin{array}{|c|c|c|} \hline a_1 & b_1 & a_1 \\ \hline c_1 & & c_1 \\ \hline a_1 & b_1 & a_1 \\ \hline \end{array}\,, \qquad \mathbf{s}_2 = \begin{array}{|c|c|c|} \hline a_2 & b_2 & a_2 \\ \hline c_2 & & c_2 \\ \hline a_2 & b_2 & a_2 \\ \hline \end{array}$$

and $\mathbf{r}_2 = \begin{array}{|c|c|c|} \hline r_{11} & r_{12} & r_{13} \\ \hline r_{21} & r_{22} & r_{23} \\ \hline r_{31} & r_{32} & r_{33} \\ \hline \end{array}\,.$

By simple computation of $\mathbf{s}_1 \boxtimes \mathbf{s}_2$, we know that $a_1 + a_2 = 4$. Since $\mathbf{r} = \mathbf{s}_1 \boxtimes \mathbf{s}_2 \vee \mathbf{r}_2$, we also have that

$$3 = \max\{a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2,\ r_{22}\}\,.$$

This contradiction shows that $\mathbf{r}$ can not be decomposed into symmetric templates.

5.4 Maxpolynomials over $(\{-\infty, 0\}, \vee, +)$

When binary images are involved, the templates used in the $\boxtimes$ convolution often have values in $\{-\infty, 0\}$. The principal tool in factorization of maxpolynomials over the belt $(\{-\infty, 0\}, \vee, +)$ is factoring by grouping. Here are three special cases when factoring by grouping is easily done.

Theorem 5.4.1. *Let k be any real number. If*

$$P(x, y) = \bigvee_{j=m}^{m+n} 0 + jx + ky$$

is a maxpolynomial in two variables, then $P(x, y) = (mx + ky) + n(0 \vee (0 + x))$.

Proof:

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + ky$$

$$= 0 + mx + ky \vee (0 + (m+1)x + ky) \vee \cdots$$

$$\vee (0 + (m+n)x + ky)$$

$$= mx + ky + (0 \vee (0+x) \vee \cdots \vee (0+nx))$$

$$= mx + ky + n(0 \vee (0+x)).$$

Q.E.D.

Theorem 5.4.2. *Let k be any real number. If $P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + (j+k)y$, is a maxpolynomial in two variables, then $P(x,y) = mx + (m+k)y + n(0 \vee (0+x+y))$.*

Proof:

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + (j+k)y$$

$$= 0 + mx + (m+k)y \vee (0 + (m+1)x + (m+1+i)y) \vee \cdots$$

$$\vee (0 + (m+n)x + (m+n+k)y)$$

$$= mx + (m+k)y + (0 \vee (0+x+y) \vee \cdots \vee (0+nx+ny))$$

$$= mx + (m+k)y + n(0 \vee (0+x+y)).$$

Q.E.D.

Theorem 5.4.3. *Let k be any real number. If $P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + (k-j)y$ is a maxpolynomial in two variables, then $P(x,y) = mx + (k-m-n)y + n((0+y) \vee (0+x))$.*

Proof:

$$P = \bigvee_{j=m}^{m+n} 0 + jx + (k-j)y$$

$$= (0 + mx + (k-m)y) \vee (0 + (m+1)x + (k-m-1)y) \vee \cdots$$

$$\vee (0 + (m+n)x + (k-m-n)y)$$

$$= mx + (k-m-n)y + ((0+ny) \vee (0+x+(n-1)y \vee \cdots \vee (0+nx)))$$

$$= mx + (k-m-n)y + n((0+y) \vee (0+x)) .$$

Q.E.D.

Although there are many other cases when factoring by grouping can be applied to reduce a maxpolynomial, these three cases play a special role in the decomposition of a certain class of convex binary templates.

Let $X \subset \mathbf{Z} \times \mathbf{Z}$. Define its convex hull, $C(X)$, as the intersection of the half planes, $H(\alpha, k)$, which contain X;

$$C(X) = \bigcap_{\alpha,k} \{H(\alpha,k) \,:\, H(\alpha,k) \supset X\} .$$

Definition. We say that X is a *convex set in* $\mathbf{Z} \times \mathbf{Z}$, when it is identical with its convex hull. Note that this definition is identical to the following when X is bounded: Let $x_i \in X$ and integers $\lambda_i \geq 0$ are such that $\sum \lambda_i = 1$, then since $x = \sum \lambda_i x_i$, $x \in X$ if and only if X is a convex set. This second approach is known as the barycentric approach.

Definition. A *restricted convex shape* is defined as a convex 4–connected component whose convex hull has boundary lines oriented only at angles 0°, 45°, 90° and 135° with respect to the positive x–axis [1].

Definition. We say that a template is a *convex (or restricted convex) template*, if its support is a convex (or restricted convex) subset of X.

If t is a restricted convex template, then its support forms a polygon in $\mathbf{R^2}$ with at most eight sides. A maxpolynomial may be associated with each of those eight sides.

Theorem 5.4.4. *A set of eight maxpolynomials corresponds to the boundary of a restricted convex template if and only if there are two of the form*

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + (k-j)y \ ,$$

two of the form

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + (j+i)y \ ,$$

two of the form

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + jx + ky \ ,$$

two of the form

$$P(x,y) = \bigvee_{j=m}^{m+n} 0 + kx + jy \ ,$$

and each one has its first term and last term in common with another maxpolynomial in the set.

Proof: Each of the polynomials represents two of the possible sides and every side shares two vertices.

Q.E.D.

In the case of a convolution of binary templates, the effects of the boundary maxpolynomials on the interior is no longer a concern. Hence, Proposition 5.3.13 may be strengthened in the following way:

Theorem 5.4.5. *Suppose that* $\mathbf{t}$ *is a restricted convex template and* $A^1(x,y),\ A^2(x,y),...,A^8(x,y)$ *correspond to a counterclockwise representation of the boundary of* $\mathbf{t}$*, where any* $A^i(x,y)$ *could be a monomial. The template* $\mathbf{t}$ *is reducible into the convolution of two restricted convex templates if and only if there exists factorizations of* $A^1(x,y),\ A^2(x,y),...,A^8(x,y)$*,*

$$A^1 = A_1^1 + A_2^1$$

$$A^2 = A_1^2 + A_1^2$$

$$\vdots$$

$$A_1^8 = A_1^8 + A_1^8$$

such that $A_1^1(x,y), A_1^2(x,y),..., A_1^8(x,y)$ *corresponds to a counterclockwise representation of the boundary of a restricted convex template.*

Proof: We have already proved one direction in theorem 5.3.13. Now suppose that such a factorization of $A^1(x,y),\ A^2(x,y),...,A^8(x,y)$ exists.

We know that each of the A_2^i are of the correct form since we know that the form of the factors of the A^i are of the correct form.

All that remains to show is that $A_2^1(x,y),\ A_2^2(x,y),...,A_2^8(x,y)$ corresponds to a counterclockwise representation of the boundary of a restricted convex template. This is equivalent to showing that if $A^i(x,y),\ A^j(x,y)$ have a common term then $A_2^i(x,y), A_2^j(x,y)$ have a common term. Suppose that $A^i(x,y), A^j(x,y)$ are adjacent. Let the common term of $A^i(x,y),\ A^j(x,y)$ be denoted by α and that of $A_1^1(x,y),\ A_1^2(x,y)$

be denoted by β. Consider α/β this term is in both $A^i/A^i_1 = A^i_2$ and $A^j/A^j_1 = A^j_2$. Hence it is a common term for them.

Thus, both $A^1_2(x,y),\ A^2_2(x,y),...,A^8_2(x,y)$ and $A^1_2(x,y),\ A^2_2(x,y),...,A^8_2(x,y)$ correspond to a counterclockwise representation of the boundary of a restricted convex template.

It is well known that the convolution of two restricted convex template is again a restricted convex template. If such a factorization exists then the maxpolynomials will give the correct boundary. And since this will be the boundary of a restricted convex template the proof is done.

Q.E.D.

There is an important note to keep in mind when applying theorem 5.4.5. When looking for a factorization of a boundary maxpolynomial we may only be looking for a monomial and that monomial may be 0.

We have now proved the following theorem.

Theorem 5.4.6. *Factoring by grouping can be used to decompose a restricted convex template into a combination of irreducible templates.*

Proof: By Theorem 5.4.5 we only need consider the boundary maxpolynomials and by Theorem 5.4.4 we know their form. Theorems 5.4.1, 5.4.2, and 5.4.3 show how factoring by grouping can be applied to these forms.

Q.E.D.

CHAPTER 6
RANK BASED MATRIX DECOMPOSITION

6.1 Introduction

Another method of template decomposition is based on matrix analysis. A rectangular shift-invariant template can be represented as a matrix. This representation of a two-dimensional rectangular shift-invariant template is achieved by letting the matrix entries, a_{ij}, be defined by $a_{ij} = \mathbf{t}_{(0,0)}(i,j)$ for all $(i,j) \in R(\mathbf{t}_{(0,0)})$, where $R(\mathbf{t}_{\mathbf{y}})$ is defined in Chapter 3. This matrix representation of the template is called the *centered weight matrix associated with* $\mathbf{t}$. By representing templates in this way, we get a one-to-one correspondence between shift-invariant templates and these matrices [24].

An image algebra computation of $\boxtimes$ involves the operations $\vee$ and $+$. Hence, the usual matrix operations do not suffice for template decomposition. Instead, one must consider minimax matrix operations.

The Ph.D. dissertation by J. Davidson showed that minimax algebra can be embedded into image algebra [25]. An important implication of this embedding is that all the tools of minimax algebra are directly applicable to solving problems in image processing whenever any image algebra operation isomorphic or dual to $\boxtimes$ is used.

In the setting of linear algebra, D. O'Leary showed that if a 5×5 matrix has either rank 1 or all of its nonzero terms are on a single diagonal, then it can be factored into the product of two 3×3 matrices [11]. Z. Manseur and D. Wilson reduced the number of factors implied by O'Leary's result for the decomposition of an arbitrary matrix by using polynomial methods [10]. J. Davidson studied some nonlinear matrix decompositions

based on minimax algebra [12]. However, the work of Davidson did not utilize the rank of a matrix. The Goal of Section 6.3 is to prove a rank based decomposition in terms of minimax algebra.

Two common belts used in the $\boxtimes$ convolution are $(\mathbf{R}_{-\infty}, \vee, +)$ and $(\{-\infty, 0\}, \vee, +)$. In the second section, we shall extend an arbitrary belt to create a bounded lattice-ordered group. Since $(\mathbf{R}_{-\infty}, \vee, +)$ and $(\{-\infty, 0\}, \vee, +)$ are commutative, many of the theorems of Cuninghame-Green are only stated for commutative belts and commutative bounded lattice-ordered groups [16].

The presentation of the definition of the rank of a matrix as defined by Cuninghame-Green requires several preliminary definitions and theorems [16]. If one were to read the definition of rank without referring to the associated theorems, one would have the impression that the definition is too limited to encompass the most general of cases, especially with regard to matrix decompositions. However, the main decomposition method presented only depends on the number of dependent columns in a matrix. Since the definition of rank is more restrictive than that of independence, rank based decompositions follow as a corollary to the main technique.

6.2 Basic Definitions

Let $(\mathbf{F}, \vee, *)$ be a division belt. We now progressively extend $(\mathbf{F}, \vee, *)$ as follows. First, we introduce the dual to $\vee$ by defining for all $x, y \in \mathbf{F}$,

$$x \wedge y = \left(x^{-1} \vee y^{-1}\right)^{-1}.$$

So then, $\mathbf{F}$ becomes a lattice ordered group, or l-group.

Next, adjoin universal bounds to $\mathbf{F}$, The elements $+\infty$ and $-\infty$ are the adjoined elements and the result is denoted by $\mathbf{F}_{\pm\infty}$.

The group operation $*$ is extended in the following manner. If $x, y \in \mathbf{F}$, then $x * y$ is already defined. Let $*' = *$ be the self-dual multiplication on elements of $\mathbf{F}$, that is,

$$x *' y = x * y \text{ for all } x, y \in \mathbf{F}.$$

Otherwise, define for all $x \in \mathbf{F}$,

$$x * -\infty = -\infty * x = -\infty$$

$$x * \infty = \infty * x = \infty$$

$$x *' -\infty = -\infty *' x = -\infty$$

$$x *' \infty = \infty *' x = \infty$$

$$(-\infty) * \infty = \infty * (-\infty) = -\infty$$

$$(-\infty) *' \infty = \infty *' (-\infty) = \infty .$$

Hence, the element $-\infty$ acts as a null element in the system $(\mathbf{F}_{\pm\infty}, \vee, *)$ and the element $+\infty$ acts as a null element in the system $(\mathbf{F}_{\pm\infty}, \wedge, *)$. The resultant structure $(\mathbf{F}_{\pm\infty}, \vee, \wedge, *, *')$ is called a bounded lattice-ordered group, or bounded l-group. We refer to $\mathbf{F}$ as the group of the bounded l-group $(\mathbf{F}_{\pm\infty}, \vee, \wedge, *, *')$. Reference to $\mathbf{F}_{\pm\infty}$ as a bounded l-group shall be with respect to $(\mathbf{F}_{\pm\infty}, \vee, \wedge, *, *')$.

Two familiar examples of bounded l-groups are $(\mathbf{R}_{\pm\infty}, \vee, \wedge, +, +')$ and $(\mathbf{R}^{\geq 0}_{\pm\infty}, \vee, \wedge, \times, \times')$. Note that $(\mathbf{R}, \vee, \wedge, +)$ is isomorphic to $(\mathbf{R}^{>0}, \vee, \wedge, \times)$ both as a group and as a lattice, and hence their extensions to l-groups will be isomorphic as well.

In recent years, lattice based matrix operations have found widespread applications in engineering sciences. In these applications, the usual matrix operations of addition and multiplication are replaced by corresponding lattice operations. For example, let

$(\mathbf{F}_{\pm\infty}, \vee, *)$ be a bounded l-group and $A = (a_{ij})$, $B = (b_{ij})$ two $m \times n$ matrices with entries in $\mathbf{F}_{\pm\infty}$.

Definition. The *point-wise maximum*, $A \vee B$ of A and B, is the $m \times n$ matrix C defined by

$$A \vee B = C, \text{ where } c_{ij} = a_{ij} \vee b_{ij} .$$

Suppose that A is $m \times p$ and B is $p \times n$.

Definition. The *product* of A and B, denoted by $A * B$, is the $m \times n$ matrix $C = A * B$, where

$$c_{ij} = \bigvee_{k=1}^{p} (a_{ik} * b_{kj}) .$$

Definition. The *dual product* of A and B, denoted by $A *' B$, is the $m \times n$ matrix $C = A *' B$, where

$$c_{ij} = \bigwedge_{k=1}^{p} (a_{ik} *' b_{kj}) .$$

The set of all $m \times n$ matrices over $\mathbf{F}_{\pm\infty}$ will be denoted by $\mathbf{M}_{mn}$.

Recall from the theory of probability that a row-stochastic matrix is a (nonnegative) matrix in which the sum of the elements in each row is unity. A column-stochastic matrix has the sum of the elements in each column equal to unity, and a doubly stochastic matrix is both row and column-stochastic.

Let $(\mathbf{F}_{\pm\infty}, \vee, \wedge, \times, \times')$ be a belt with duality and $(\sigma_\emptyset, \vee, \wedge, \times, \times')$ a sub-belt of $\mathbf{F}_{\pm\infty}$ with duality. We shall say that a finite subset $S \subset \mathbf{F}$ is $\sigma_\emptyset$–astic, if it is true that

$$\bigvee_{x \in S} x \in \sigma_\emptyset .$$

Let $1_{\mathbf{F}}$ be the identity with respect to $*$. If $\sigma_{\emptyset}$ is just $1_{\mathbf{F}}$, then a $\sigma_{\emptyset}$–astic set satisfies:

$$\bigvee_{x \in S} x = 1_{\mathbf{F}} .$$

A matrix over $\mathbf{F}_{\pm\infty}$ will be called row-$\sigma_{\emptyset}$–astic (respectively column-$\sigma_{\emptyset}$–astic, or doubly $\sigma_{\emptyset}$–astic) if the elements in each row (respectively each column, or each row and column) form a $\sigma_{\emptyset}$–astic set.

Definition. A square matrix $A \in \mathbf{M}_{nn}$ is *strictly doubly* $1_{\mathbf{F}}$*–astic*, if it satisfies the following two requirements.

(i) $A_{ij} \leq 1_{\mathbf{F}}$ for all $i = 1, ..., n$ and $j = 1, ..., n$.

(ii) On each row and on each column of A, we can find one and only one element equal to $1_{\mathbf{F}}$.

If $A \in \mathbf{M}_{mn}$, then A has n columns, $a_{i1}, a_{i2}, ..., a_{in}$, each of which is an m–tuple. For notational purposes, let $a(j) = a_{ij}$, $i = 1, 2, ..., m$, so that $a(j)$ is the j–th column.

Let $X \in \mathbf{M}_{n1}$ and $B \in \mathbf{M}_{m1}$. The equation $A * X = B$ may then be written,

$$\left(\bigvee_{j=1}^{n} a(j) * x_j \right) = B .$$

Definition. The relation, $\left(\bigvee_{j=1}^{n} a(j) * x_j \right) = B$, expresses the *linear dependence (over* $\mathbf{F}$*) of* B *on* $a(j)$. We shall also say that B is a *linear combination of* $a(1), ..., a(n)$, (even when $n = 1$).

Let $\mathbf{F}_{\pm\infty}$ be a bounded l-group. Suppose that we are given m–tuples, $a(j)$, $j = 1, ..., n$ and we wish to determine, for each of them, whether or not it is linearly dependent on the other $(n - 1)$ m–tuples. The next theorem gives a convenient mechanical procedure.

Let $A \in \mathbf{M}_{mn}$ be the matrix having $a(j)$ as its j–th column. Let A^* be defined by $(A^*)_{ij} = (A_{ji})^*$, where $(A_{ji})^*$ is the conjugate of A_{ji} as defined in Chapter 2. Define a matrix $\mathcal{A} \in \mathbf{M}_{mn}$ as follows. Let

$$\mathcal{A}_{ii} = -\infty\,,\ i = 1, ..., n,$$

and

$$\mathcal{A}_{ij} = \left(A^* *' A\right)_{ij}, i = 1, ..., n\,,\ j = 1, ..., m,\ i \neq j.$$

In other words, $\mathcal{A}$ is the matrix $A^* *' A$ with its diagonal elements overwritten by $-\infty$. We now compare each column of A with the corresponding column of $A \in \mathbf{M}_{mn}$ and make use of the following theorem.

Theorem 6.2.1. (Cuninghame-Green, Theorem 16.2) *Let* $\mathbf{F}$ *be a commutative bounded l-group. Let the matrix* $A \in \mathbf{M}_{mn}$ *have columns* $a(j) \in \mathbf{M}_{m1}\,, j = 1, ..., n \geq 2$, *not necessarily all different. For each* $j = 1, ..., n$, *the* j*–th column of* $A * \mathcal{A}$ *is identical with* $a(j)$ *if and only if* $a(j)$ *is linearly dependent on the other columns of* A. *The elements of the* j*– th column of* $\mathcal{A}$ *then give suitable coefficients to express the linear dependence.*

Note that the proof of this theorem shows that if the d–th column is dependent, then $\mathcal{A}_{jd}$ is the coefficient corresponding to the column $a(j)$.

Example. Let

$$A = \begin{pmatrix} 1 & 3 & 2 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 5 & 5 & 3 \end{pmatrix}.$$

To compute $\mathcal{A}$, first find

$$A^* *' A = \begin{pmatrix} -1 & -3 & -2 \\ -3 & -4 & -5 \\ -2 & -2 & -5 \\ -3 & -1 & -3 \end{pmatrix} *' \begin{pmatrix} 1 & 3 & 2 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 5 & 5 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & -1 & -2 \\ -3 & 0 & -2 & -3 \\ -3 & 0 & 0 & -2 \\ -3 & 0 & -1 & 0 \end{pmatrix}.$$

Hence,

$$\mathcal{A} = \begin{pmatrix} -\infty & 1 & -1 & -2 \\ -3 & -\infty & -2 & -3 \\ -3 & 0 & -\infty & -2 \\ -2 & 0 & -1 & -\infty \end{pmatrix}$$

and

$$A * \mathcal{A} = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 1 & 4 & 2 & 1 \\ 2 & 5 & 3 & 3 \end{pmatrix}.$$

Applying Theorem 6.2.1, it can be seen that the second column is linearly dependent on the other three. However, note that column one is not linearly dependent on the other columns. This is a major difference between conventional linear algebra and minimax algebra. In conventional linear algebra, the equation

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = b$$

would also imply that a_j is a linearly dependent on $\{b\} \cup \{a_i\}_{i=1, i \neq j}^{n}$.

There are situations, particularly if the matrix is symmetric, that minimax linear dependence mimics conventional linear algebra in this regard. In those situations, one way to effectively apply the methods of Theorem 6.2.1 is to analyze the columns inductively. If a linearly dependent column is found, disregard it in the next step of the analysis. If it is not dependent keep it in the next step. So, if $a(j),\ j = 1, ..., n-1$ are not linearly dependent, apply Theorem 6.2.1 to $a(j),\ j = 1, ..., n$. If $a(n)$ is dependent on $a(j),\ j = 1, ..., n-1$, then next apply the theorem to $a(j),\ j = 1, ..., n-1, n+1$,

leaving out $a(n)$. If $a(n)$ is not dependent on $a(j)$, $j = 1, ..., n-1$, then next apply the theorem to $a(j)$, $j = 1, ..., n+1$,, including $a(n)$.

The purpose of the next two theorems is to show some of the anomalies associated with linear dependence as it may lead to the definition of rank.

Theorem 6.2.2. (Cuninghame-Green, Theorem 16.4) *Suppose that* $\mathbf{F}$ *is a commutative bounded l-group other than* $(\{-\infty, 0, +\infty\}, \vee, \wedge, +, +')$. *Let* $m > 2$ *and* $k > 1$ *be arbitrary integers. We can always find* k *finite* m*–tuples, no one of which is linearly dependent on the others.*

Theorem 6.2.3. (Cuninghame-Green, Theorem 16.5) *Suppose that* $\mathbf{F} = (\{-\infty, 0, +\infty\}, \vee, \wedge, +, +')$. *Let* $m > 2$. *We can always find (at least)* $m^2 - m$ m*–tuples, no one of which is linearly dependent on the others.*

In conventional linear algebra, a number of different, but logically equivalent, definitions are possible of the notion of linear independence of a set of elements of a vector space. However, Cuninghame-Green formulated analogous minimax algebra definitions of various alternative forms of linear independence of elements of a band-space, and showed that they are not logically equivalent, although certain logical implications may be demonstrated among them [16]. These considerations led to the following definition.

Definition. Let $\mathbf{F}$ be a bounded l-group and let $a(1), ..., a(k) \in \mathbf{M}_{n1}$. We shall say that $a(1), ..., a(k)$ are *strongly linearly independent*, if there is at least one finite n–tuple, $B \in \mathbf{M}_{n1}$, which has a unique expression in the form

$$B = \left(\bigvee_{r=1}^{t} a(j_r) + \lambda_{j_r} \right) \tag{1}$$

with $\lambda_{j_r} \in \mathbf{F}$, $1 \leq j_r \leq k$, $(r = 1, ..., t)$ and $j_r < j_s$ if $r < s$ $(r = 1, ..., t;\ s = 1, ..., t)$. We shall abbreviate "strongly linearly independent" by SLI.

For a given belt, $\mathbf{F}_{\pm\infty}$, define linear independence as the negation of linear dependence.

Definition. $\mathrm{a}(1), ..., \mathrm{a}(k) \in \mathbf{F}^n$ are *linearly independent* exactly when no one of them is linearly dependent on the others.

The next theorem relates the definitions of SLI and linear independence.

Theorem 6.2.4. (Cuninghame-Green, Theorem 16.10) *Let* $\mathbf{F}_{\pm\infty}$ *be a commutative bounded l-group and* $\mathrm{a}(1), ..., \mathrm{a}(k) \in \mathbf{M}_{n1}$. *For* $\mathrm{a}(1), ..., \mathrm{a}(k)$ *to be linearly independent it is sufficient, but not necessary, that* $\mathrm{a}(1), ..., \mathrm{a}(k)$ *be SLI.*

Definition. Let $\mathbf{F}_{\pm\infty}$ be any bounded l-group and let $A \in \mathbf{M}_{mn}$. Suppose that we can find r columns $(1 \leq r \leq n)$ of A, but no more, which are SLI. We shall say that A has *column-rank* equal to r. We define *row-rank* of A as the column-rank of the transpose of A.

Before proving relationships among these ranks, we need one more definition.

Definition. A given matrix $A \in \mathbf{M}_{mn}$ has $1_{\mathbf{F}}$–astic rank equal to r, if the following is true for $k = r$ but not for $k > r$.

(i) There are $X \in \mathbf{M}_{n1}$ and $Y \in \mathbf{M}_{m1}$, both finite, such that $B \in \mathbf{M}_{mn}$ is doubly $1_{\mathbf{F}}$–astic and contains a $k \times k$ strictly doubly $1_{\mathbf{F}}$–astic submatrix, where

$$B_{ij} = Y_i * A_{ij} * X_j \quad (i = 1, ..., m;\ j = 1, ..., n)$$

Theorem 6.2.5. (Cuninghame-Green, Theorem 17.7) *Let* $\mathbf{F}_{\pm\infty}$ *be a linear commutative bounded l-group with group* $\mathbf{F}$ *and let* $A \in \mathbf{M}_{mn}$ *be doubly* $1_{\mathbf{F}}$*–astic. The following statements are then equivalent.*

(i) A *has* $1_{\mathbf{F}}$*–astic rank equal to* r.

(ii) A *has column-rank equal to* r.

(iii) A *has row-rank equal to* r.

(iv) A^* *has dual column-rank equal to* r

(v) A *has dual row-rank equal to* r

In view of Theorem 6.2.5, we may (for doubly $1_{\mathbf{F}}$–astic A) simply use the expression *rank* of A.

In the foregoing results, the equality of various ranks of a matrix have been demonstrated, if they exist. We have not yet discussed whether a matrix necessarily has such ranks. The next theorem answers this question.

Theorem 6.2.6. (Cuninghame-Green, Theorem 17.9) *Let* $\mathbf{F}_{\pm\infty}$ *be linear commutative bounded l-group with group* $\mathbf{F}$ *and let* $A \in \mathbf{M}_{mn}$. *There exists an integer* r *such that* A *has* $1_{\mathbf{F}}$*–astic rank* r *if and only if* A *is doubly* $\mathbf{F}$*–astic . The integer* r *satisfies* $1 \leq r \leq$ *min*$\{m, n\}$.

6.3 Matrix Decomposition

We begin with the weaker condition of linear independence.

Theorem 6.3.1. *If* $A \in \mathbf{M}_{mn}$ *is a matrix with* r *linearly independent columns, then*

$$A = A_1 \vee A_2 \vee \cdots \vee A_r,$$

where each A_i *is of size* $m \times n$ *and has one linearly independent column.*

Proof: Let D denote the set of indices of the dependent columns.

For each independent $a(j)$, define A_j as follows.

Let the j–th column of A_j be $a(j)$. For each $d \in D$, let the d–th column of A_j be $\mathcal{A}_{jd} * a(j)$, where $\mathcal{A}$ is from Theorem 6.2.1. According to Theorem 6.2.1, $A = A_1 \vee A_2 \vee \cdots \vee A_r$. Since each A_i consist of a single non $-\infty$ column, $a(j)$, and $\mathcal{A}_{jd} * a(j)$, they all have one linearly independent column.

Q.E.D.

Example. Let

$$A = \begin{pmatrix} 1 & 3 & 2 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 5 & 5 & 3 \end{pmatrix}.$$

We have that

$$\mathcal{A} = \begin{pmatrix} -\infty & 1 & -1 & -2 \\ -3 & -\infty & -2 & -3 \\ -3 & 0 & -\infty & -2 \\ -2 & 0 & -1 & -\infty \end{pmatrix}$$

and

$$A * \mathcal{A} = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 1 & 4 & 2 & 1 \\ 2 & 5 & 3 & 3 \end{pmatrix}.$$

Hence, the second column is linearly dependent on the other columns. Also $\mathcal{A}_{12} = 1$, $\mathcal{A}_{32} = 0$, $\mathcal{A}_{42} = 0$. According to Theorem 6.3.1,

$$A = \begin{pmatrix} 1 & 2 & -\infty & -\infty \\ 3 & 4 & -\infty & -\infty \\ 2 & 3 & -\infty & -\infty \end{pmatrix} \vee \begin{pmatrix} -\infty & 2 & 2 & -\infty \\ -\infty & 2 & 2 & -\infty \\ -\infty & 5 & 5 & -\infty \end{pmatrix}$$

$$\vee \begin{pmatrix} -\infty & 3 & -\infty & 3 \\ -\infty & 1 & -\infty & 1 \\ -\infty & 3 & -\infty & 3 \end{pmatrix}.$$

Corollary 6.3.2. *If $A \in \mathbf{M}_{mn}$ is a matrix with rank r, then*

$$A = A_1 \vee A_2 \vee \cdots \vee A_r$$

where each A_i is of size $m \times n$ and has one linearly independent column.

Proof: If $A \in \mathrm{M}_{mn}$ is a matrix with rank r, then by Theorem 6.2.5 A has r columns which are SLI. By Theorem 6.2.4, r SLI columns implies r independent columns.

Q.E.D.

Thus, if the centered weight matrix, A, corresponding to a template, $\mathbf{t}$, has r independent columns, then we can write $\mathbf{t}$ as

$$\mathbf{t} = \mathbf{t}_1 \vee \mathbf{t}_2 \vee \cdots \vee \mathbf{t}_r,$$

where $\mathbf{t}_i$ is separable template for each $i = 1, 2, ..., n$. A separable template can then be decomposed into a row and a column template, namely $\mathbf{t}_i = \mathbf{r}_i \boxtimes \mathbf{s}_i$. Therefore,

$$\mathbf{t} = (\mathbf{r}_1 \boxtimes \mathbf{s}_1) \vee (\mathbf{r}_2 \boxtimes \mathbf{s}_2) \vee \cdots \vee (\mathbf{r}_n \boxtimes \mathbf{s}_n).$$

Example. Let

$$\mathbf{t} = \begin{array}{|c|c|c|c|c|} \hline -2 & -2 & -2 & -2 & -2 \\ \hline -2 & -1 & -1 & -1 & -2 \\ \hline -2 & -1 & 0 & -1 & -2 \\ \hline -2 & -1 & -1 & -1 & -2 \\ \hline -2 & -2 & -2 & -2 & -2 \\ \hline \end{array}.$$

The centered weight matrix corresponding to this template is

$$T = \begin{pmatrix} -2 & -2 & -2 & -2 & -2 \\ -2 & -1 & -1 & -1 & -2 \\ -2 & -1 & 0 & -1 & -2 \\ -2 & -1 & -1 & -1 & -2 \\ -2 & -2 & -2 & -2 & -2 \end{pmatrix}.$$

According to Theorem 6.3.1, we may write $T = T_1 \vee T_2 \vee T_3$, where

$$T_1 = \begin{pmatrix} -2 & -\infty & -\infty & -2 & -2 \\ -2 & -\infty & -\infty & -2 & -2 \\ -2 & -\infty & -\infty & -2 & -2 \\ -2 & -\infty & -\infty & -2 & -2 \\ -2 & -\infty & -\infty & -2 & -2 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} -\infty & -2 & -\infty & -2 & -3 \\ -\infty & -1 & -\infty & -1 & -2 \\ -\infty & -1 & -\infty & -1 & -2 \\ -\infty & -1 & -\infty & -1 & -2 \\ -\infty & -2 & -\infty & -2 & -3 \end{pmatrix},$$

and

$$T_3 = \begin{pmatrix} -\infty & -\infty & -2 & -3 & -4 \\ -\infty & -\infty & -1 & -2 & -3 \\ -\infty & -\infty & 0 & -1 & -2 \\ -\infty & -\infty & -1 & -2 & -3 \\ -\infty & -\infty & -2 & -3 & -4 \end{pmatrix}.$$

If we take $\mathbf{t}_i$ to be the template corresponding to the centered weight matrix T_i, then each $\mathbf{t}_i$ is separable. Thus, $\mathbf{t} = (\mathbf{r}_1 \boxtimes \mathbf{s}_1) \vee (\mathbf{r}_2 \boxtimes \mathbf{s}_2) \vee (\mathbf{r}_3 \boxtimes \mathbf{s}_3)$, where

$$\mathbf{r}_1 = \begin{matrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{matrix}, \quad \mathbf{r}_2 = \begin{matrix} -2 \\ -1 \\ -1 \\ -1 \\ -2 \end{matrix}, \quad \mathbf{r}_3 = \begin{matrix} -2 \\ -1 \\ 0 \\ -1 \\ -2 \end{matrix},$$

$$\mathbf{s}_1 = \begin{matrix} 0 & -\infty & -\infty & 0 & 0 \end{matrix},$$

$$\mathbf{s}_2 = \begin{matrix} -\infty & 0 & -\infty & 0 & -1 \end{matrix},$$

and $$\mathbf{s}_3 = \begin{matrix} -\infty & -\infty & 0 & -1 & -2 \end{matrix}.$$

The converse of Theorem 6.3.1 is not true. Specifically, it can be shown that if a matrix A has a decomposition in the form $A = A_1 \vee A_2$, where each A_i has one linearly independent column, it may not be true that A has two linearly independent columns. The next example shows how this can happen.

Example. If

$$A_1 = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 6 & 8 \\ 4 & 8 & 10 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -\infty & 7 & -\infty \\ -\infty & 5 & -\infty \\ -\infty & 9 & -\infty \end{pmatrix},$$

then

$$A = A_1 \vee A_2 = \begin{pmatrix} 1 & 7 & 7 \\ 2 & 6 & 8 \\ 4 & 9 & 10 \end{pmatrix}.$$

The matrix A has three linearly independent columns.

CHAPTER 7
CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

This dissertation has developed the theory of maxpolynomials. A particular emphasis has been placed on using their factorization as a method of decomposing morphological structuring elements.

The steps in the development were:

1) A definition of maxpolynomials given in terms of sequences of elements. This definition allows for the complete classification of their algebraic structure. This classification is based on existing minimax theory.

2) A counter example showed that a division algorithm does not hold for maxpolynomials. However, we developed a division procedure for the one variable case which can be applied in most practical cases.

3) The presentation of several sufficient conditions for the factorization of one variable maxpolynomials. Particular emphasis was placed on those exhibiting symmetry, due to their frequency of use in image processing.

4) The necessary and sufficient conditions under which a two variable maxpolynomials can be decomposed into two one variable maxpolynomials. The previous result in this area only applied to maxpolynomials which corresponded to rectangular templates.

5) A necessary condition for the decomposition of two-dimensional templates is the decomposition of their boundaries. The one variable techniques were extended to the two variable case. Since most template are two-dimensional, these results should be the most useful.

6) A rank based matrix decomposition in terms of minimax algebra was proven.

The following are suggestions for further research:

The primary theoretical results on polynomial factorization and irreducibility are derived from the algebraic structure on the coefficients. The theorems of Chapter 4 lead to the investigation of such possibilities for maxpolynomials. We now may consider conditions on the belt of coefficients. Do notions such as divisibility and irreducibility exist in belts? Are there properties of certain belts which aid in the factorization of maxpolynomials?

The splitting field of the real numbers is the complex numbers. Is there an extension of $(\mathbf{R}_{-\infty}, \vee, +)$ which leads to an equivalent form of the fundamental theorem of algebra? Since there is no fundamental theorem at this time, many more factorization techniques for specific maxpolynomials need to be developed.

We considered methods for decomposing two variable maxpolynomials based on their boundary. The arrangement of the boundary factorization has a substantial effect on the interior. Is there a minimal configuration for the boundary factorization? Extensions of the factorization results presented here can include algorithms to determine the interior of the decompositions so to optimize any remainder which may exist.

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BIOGRAPHICAL SKETCH

Frank Crosby was born in Detroit, Michigan on the third and worst night of the riots in 1967. He lived in Detroit until attending Alabama Agricultural and Mechanical University. At Alabama A. and M., he played both varsity tennis and soccer. He received a B.S. in Mathematics in 1989. His graduate studies began at the University of Michigan where he got an M.S. in Applied Mathematics in 1991. He began studying image algebra at the University of Florida in the Fall of 1991 and graduated with a Ph.D. in Mathematics in 1995.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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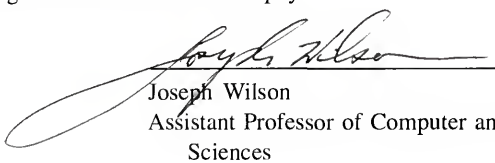

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